

A Grand $\Delta(96) \times SU(5)$ Flavour Model

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Abstract

Recent results from the Daya Bay and RENO reactor experiments have measured the smallest lepton mixing angle and found it to have a value of $\theta_{13} \approx 9^\circ$. This result presents a new challenge for the existing paradigms of discrete flavour symmetries which attempt to describe all quark and lepton masses and mixing angles. Here we propose a Supersymmetric Grand Unified Theory of Flavour based on $\Delta(96) \times SU(5)$, together with a $U(1) \times Z_3$ symmetry, including a full discussion of $\Delta(96)$ in a convenient basis. The Grand $\Delta(96) \times SU(5)$ Flavour Model relates the quark mixing angles and masses in the form of the Gatto-Sartori-Tonin relation and realises the Georgi-Jarlskog mass relations between the charged leptons and down-type quarks. We predict a Bi-trimaximal (not Tri-bimaximal) form of neutrino mixing matrix, which, after including charged lepton corrections with zero phase, leads to the following GUT scale predictions for the atmospheric, solar, and reactor mixing angles: $\theta_{23} \approx 36.9^\circ$, $\theta_{12} \approx 32.7^\circ$ and $\theta_{13} \approx 9.6^\circ$, in good agreement with recent global fits, and a zero Dirac CP phase $\delta \approx 0$.

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1 Introduction

It is one of the goals of theories of particle physics beyond the Standard Model to predict quark and lepton masses and mixings, or at least to relate them. While the quark mixing angles are known to all be rather small, by contrast two of the lepton mixing angles, the atmospheric angle θ_{23} and the solar angle θ_{12} , are identified as being rather large [1]. Until recently the remaining reactor angle θ_{13} was unmeasured. Direct evidence for θ_{13} was first provided by T2K, MINOS and Double Chooz [2–4]. Recently Daya Bay [5], RENO [6], and Double Chooz [7] Collaborations have measured $\sin^2(2\theta_{13})$:

$$\begin{aligned} \text{Daya Bay:} \quad & \sin^2(2\theta_{13}) = .089 \pm .011(\text{stat.}) \pm .005(\text{syst.}) , \\ \text{RENO:} \quad & \sin^2(2\theta_{13}) = .113 \pm .013(\text{stat.}) \pm .019(\text{syst.}) , \\ \text{Double Chooz:} \quad & \sin^2(2\theta_{13}) = .109 \pm .030(\text{stat.}) \pm .025(\text{syst.}) . \end{aligned} \tag{1}$$

From a theoretical or model building point of view, one significance of this measurement is that it excludes the tri-bimaximal (TB) lepton mixing pattern [8] in which the atmospheric angle is maximal, the reactor angle vanishes, and the solar mixing angle is approximately 35.3° . When comparing global fits to TB mixing it is convenient to express the solar, atmospheric and reactor angles in terms of deviation parameters (s , a and r) from TB mixing [9]:

$$\sin \theta_{12} = \frac{1}{\sqrt{3}}(1 + s), \quad \sin \theta_{23} = \frac{1}{\sqrt{2}}(1 + a), \quad \sin \theta_{13} = \frac{r}{\sqrt{2}}. \tag{2}$$

The global fit in [10] (which has been updated to include the data released at the Neutrino 2012 Conference; see also [11]) yields the 1σ ranges for the TB deviation parameters:

$$-0.066 \leq s \leq -0.013, \quad -0.146 \leq a \leq -0.094, \quad 0.208 \leq r \leq 0.231, \tag{3}$$

assuming a normal neutrino mass ordering. As well as showing that TB is excluded by the reactor angle being non-zero, Eq. (3) shows a preference for the atmospheric angle to be below its maximal value and also a slight preference for the solar angle to be below its tri-maximal value.

As a result of the rapidly changing landscape of neutrino mixing parameters, many models based on discrete family symmetry (see [12] and references therein) which were proposed initially to account for TB mixing are now either excluded, or have been subjected to modification [13, 14]. A promising new approach has emerged based on a new discrete family symmetry, namely $\Delta(96)$, which is capable of predicting the value of the reactor mixing angle [15, 16]. It was found that $\Delta(96)$ can lead to two alternative mixing patterns, related by exchange of the lower two rows of the PMNS mixing matrix, depending on the particular choice of Klein symmetry $Z_2^S \times Z_2^U$ that is respected by the neutrino mass matrix without re-ordering the three generations of lepton doublets. In this paper we shall select the following Klein group generators for the triplet representation $\mathbf{3}$ of $\Delta(96)$ (see Appendix A):

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad U = \frac{1}{3} \begin{pmatrix} -1 + \sqrt{3} & -1 - \sqrt{3} & -1 \\ -1 - \sqrt{3} & -1 & -1 + \sqrt{3} \\ -1 & -1 + \sqrt{3} & -1 - \sqrt{3} \end{pmatrix}, \tag{4}$$

where $SU = US$ and $S^2 = U^2 = I$. The breaking of the $\Delta(96)$ family symmetry to this particular Klein symmetry in the neutrino sector will be achieved through a set of scalar fields (flavons) coupling to neutrino mass terms. These flavons will obtain vacuum expectation values (vevs) that will only leave invariant the $Z_2^S \times Z_2^U$ subgroup contained in $\Delta(96)$.

Assuming the above choice of Klein symmetry in the neutrino sector, together with a diagonal charged lepton mass matrix, $\Delta(96)$ predicts the following PMNS mixing matrix:

$$U_{\text{BT}} = \begin{pmatrix} a_+ & \frac{1}{\sqrt{3}} & a_- \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} P, \quad (5)$$

where $a_{\pm} = (1 \pm \frac{1}{\sqrt{3}})/2$ and P is the usual diagonal matrix containing the Majorana phases. We shall refer to this as ‘‘Bi-trimaximal’’ (BT) mixing due to the distinctive St George’s cross feature of the middle row and column being of the tri-maximal form.¹ This leads to the following predictions:

$$\begin{aligned} \sin \theta_{12} = \sin \theta_{23} &= \sqrt{\frac{8-2\sqrt{3}}{13}} \approx 0.591 & (\theta_{12} = \theta_{23} \approx 36.2^\circ), \\ \sin \theta_{13} &= a_- \approx 0.211 & (\theta_{13} \approx 12.2^\circ). \end{aligned} \quad (6)$$

In terms of the TB deviations, the BT mixing pattern predicts:

$$s \approx 0.023, \quad a \approx -0.165, \quad r \approx 0.299, \quad (7)$$

which all fall outside the 1σ ranges for these parameters in Eq. (3). This motivates going beyond the simple models of leptons proposed so far [16], and in particular to Grand Unified models where the charged lepton mass matrix is only approximately diagonal and the resulting charged lepton mixing corrections will slightly modify the above predictions, bringing them into agreement with the global fits.

The purpose of the present paper is to construct the first $\Delta(96) \times SU(5)$ Grand Unified Theory (GUT) of Flavour in a Supersymmetric (SUSY) context, where the model leads to BT neutrino mixing, modified by charged lepton corrections. The model relates the quark mixing angles and masses in the form of the Gatto-Sartori-Tonin (GST) [17] relation and realises the Georgi-Jarlskog (GJ) [18] mass relations between the charged leptons and down-type quarks. In order to do this we have to develop the group theory of $\Delta(96)$ beyond that which appears in [16] where two models of leptons were proposed based on $\Delta(96)$. The reason is that in [16] the charged lepton mass eigenvalues required re-ordering before a physical interpretation could be achieved. However such a re-ordering is not convenient from the point of view of GUT theories, since the swapping of rows and columns can cloud the hierarchical structures that one wishes to achieve. Here we calculate the Clebsch-Gordan (CG) coefficients in a suitable basis right from the start. We also emphasise the use of the S , T , and U generators to draw analogy to previous models based on S_4 (A_4). The lengthy but necessary group theoretical aspect of this work is relegated to the Appendix.

¹An alternative choice of Klein symmetry leads to the second and third rows of Eq. (5) being interchanged [15], and hence an atmospheric angle in the second octant, somewhat disfavoured by the recent global fits.

The remainder of this work is as follows. In Section 2, the $\Delta(96) \times SU(5)$ SUSY GUT is introduced by defining its fields and their transformation properties under $\Delta(96)$, $SU(5)$, as well as an additional $U(1)$ and Z_3 symmetry. The resulting mass matrices are calculated after flavour and electroweak symmetry breaking to reveal the quark and lepton mass matrices from which the predictions for the quark and lepton mixing parameters are obtained. In Section 3 the vacuum alignments assumed in Section 2 are justified by constructing a flavon potential and deriving the flavon alignments from minimisation conditions. In Section 4, possible subleading corrections to the leading order (LO) predictions are discussed. Section 5 concludes the paper. The group theory of $\Delta(96)$ is elicited in Appendix A by constructing its generators, character table, and corresponding CG coefficients.

2 The $\Delta(96) \times SU(5)$ Model

2.1 Fields, Symmetries and Yukawa Operators

In this section, we present an $SU(5)$ GUT, endowed with a $\Delta(96)$ flavour symmetry (the relevant group theory of $\Delta(96)$ can be found in Appendix A). We follow closely the logic of the $S_4 \times SU(5)$ GUT presented in Ref. [20]. Therefore, we assign the $\bar{\mathbf{5}}$ matter fields of $SU(5)$ to the triplet representation $\mathbf{3}$ of $\Delta(96)$ and call them F . The first two particle generations' $\mathbf{10}$ dimensional matter fields are assigned to the doublet $\mathbf{2}$ of $\Delta(96)$, and the third generation's $\mathbf{10}$ dimensional matter field is assigned to the singlet $\mathbf{1}$ (call these T and T_3 , respectively). Also, right-handed neutrino fields, N , are added that transform as a singlet under $SU(5)$ and a $\bar{\mathbf{3}}$ under $\Delta(96)$.

In addition to the above matter fields, the GUT Higgs fields, H_5 , $H_{\bar{5}}$, and $H_{\overline{45}}$ are added and are all singlets under the $\Delta(96)$ flavour symmetry. Notice that the MSSM Higgs doublets H_u and H_d (whose vevs are related by $v_u/v_d = \tan \beta$) originate from H_5 and a linear combination of $H_{\bar{5}}$ and $H_{\overline{45}}$, respectively.² We have added the $H_{\overline{45}}$ to obtain the GJ relations between the charged lepton and down quark masses. With this minimal field content transforming under $SU(5)$ and $\Delta(96)$, the only Yukawa terms in the superpotential which are allowed are FNH_5 , $T_3T_3H_5$, and $TT H_5$. These renormalisable terms correspond to a Dirac neutrino mass, a top quark mass and masses for up and charm quarks, respectively. The former term is problematic because it produces a degenerate ~ 100 GeV neutrino mass spectrum (if Yukawa couplings are not tuned accordingly) with no mixing. The latter term is problematic because it yields degenerate masses for the up and the charm quark, and provides little explanation for the observed patterns of quark mixing.

In view of the above phenomenological requirements, it is necessary to introduce new symmetries which forbid any un-phenomenological operator, as well as to add more fields (*i.e.* flavon fields) which couple to the existing matter fields to fix the problematic predictions mentioned above. The additional flavon fields, denoted by Φ_ρ^f , are all $SU(5)$ gauge singlets and charged under the family symmetry. Here f represents the fermion sector that the

²The orthogonal linear combination of $H_{\bar{5}}$ and $H_{\overline{45}}$ is assumed to decouple from the low-energy theory by obtaining a GUT scale mass. Note that the $SU(5)$ Higgs potential will not be studied because the aim of this work is to construct a theory of flavour. Therefore, minimising the full GUT Higgs potential is beyond the scope of this work.

Field	T_3	T	F	N	$H_{5,\bar{5}}$	$H_{\overline{45}}$	Φ_2^u	$\bar{\Phi}_2^u$	Φ_3^d	$\bar{\Phi}_3^d$	Φ_2^d	$\Phi_{3'}^\nu$	$\Phi_{3'}^\nu$	Φ_3^ν
$SU(5)$	10	10	$\bar{5}$	1	$5, \bar{5}$	$\overline{45}$	1	1	1	1	1	1	1	1
$\Delta(96)$	1	2	3	$\bar{3}$	1	1	2	2	$\bar{3}$	$\bar{3}$	2	$\bar{3}'$	$\tilde{3}'$	$\tilde{3}$
$U(1)$	0	x	y	$-y$	0	z	$-2x$	0	$-y$	$-x - y - 2z$	z	$2y$	$2y$	w
Z_3	1	1	ω^2	ω	$1, \omega$	ω	1	1	1	1	1	ω	ω	ω

Table 1: The field content of the $\Delta(96) \times SU(5) \times U(1) \times Z_3$ model. x, y, z , and w are integers, while $\omega = e^{\frac{2\pi i}{3}}$.

flavon field couples to and can be either u , d , or ν , and ρ labels the representation of $\Delta(96)$ that the flavon field transforms under. Thus, ρ can take any of the ten values of irreducible representations of $\Delta(96)$ (*i.e.* **1**, **1'**, **2**, **3**, **$\bar{3}$** , **$\tilde{3}$** , **$\bar{3}'$** , **$\tilde{3}'$** , **$\bar{6}$**). In addition to these flavon fields, a global $U(1)$ symmetry must be introduced to prevent a flavon field associated with one fermionic sector contaminating another sector, as well as preventing the leading order $TT H_5$ term. This $U(1)$ symmetry will not be gauged in order to avoid the constraints of anomaly cancellation. However, it will be spontaneously broken by the flavon fields acquiring vevs in our model. This will lead to a massless Goldstone boson, unless the $U(1)$ symmetry is also explicitly broken. For this reason, the $U(1)$ symmetry is assumed to be explicitly broken in the hidden sector of the theory so that the Goldstone boson becomes a Pseudo-Goldstone boson with mass around 1 TeV. Also, an additional Z_3 symmetry will be added to prevent the proliferation of problematic terms in the flavon superpotential (see Section 3 for further explanation). A summary of the model's field content and its transformation properties under $\Delta(96) \times SU(5) \times U(1) \times Z_3$ can be found in Table 1.

Using the transformation properties given in Table 1, the lowest order Yukawa operators invariant under all symmetries can be constructed. The resulting terms contributing to the up quark mass matrix read

$$y_u T_3 T_3 H_5 + y'_u \frac{1}{M} TT \Phi_2^u H_5 + y''_u \frac{1}{M^2} TT \Phi_2^u \bar{\Phi}_2^u H_5, \quad (8)$$

where M is a generic messenger scale (around the GUT scale) common to all higher-dimensional operators and (presumably) order one coupling constants are denoted by y_u , y'_u , y''_u . If not explicitly stated, all contractions of flavon fields are taken into account. With the up sector written, the charges in Table 1 can be used to write down the Yukawa operators for the down quarks and charged leptons:

$$y_d \frac{1}{M} F T_3 \Phi_3^d H_{\bar{5}} + y'_d \frac{1}{M^2} (F \bar{\Phi}_3^d)_1 (T \Phi_2^d)_1 H_{\overline{45}} + y''_d \frac{1}{M^3} (F \Phi_2^d \Phi_2^d)_3 (T \bar{\Phi}_3^d)_{\bar{3}} H_{\bar{5}}, \quad (9)$$

where the subscripts on the parentheses denote the specific contraction being taken from the tensor product contained inside of the parentheses, and we have again denoted order one constants by y_d , y'_d , y''_d . Obviously, there are other contractions involving the fields in parentheses, which are not written down. Thus, a specific set of mediator fields existing above the messenger scale, M , which will select out the desired contractions in Eq. (9) is assumed. For a similar example of how this can be realised in an ultraviolet-completed model, see e.g. [20, 21]. Notice that the second term in Eq. (9) gives rise to the GJ relation (when

suitable vevs are applied) which provides phenomenologically acceptable charged lepton and down quark masses, when extrapolated to low-energy scales. When the flavon fields in the third term of Eq. (9) assume suitable vevs, the GST relation is fulfilled because of equal (12) and (21) entries and a (11) entry equal to zero, at this order. This relation gives rise to the successful prediction of the ratio of the down and strange quark masses to the Cabibbo angle (*i.e.* $\theta_{12}^q \approx \theta_{12}^d \approx \sqrt{m_d/m_s}$). Now that the lowest order quark and charged lepton Yukawa operators are written, attention is turned to the neutrino sector which has the following leading terms, constructed from the transformation properties in Table 1:

$$y_D F N H_5 + \bar{y}_M N N \Phi_3^\nu + \tilde{y}_M N N \Phi_3^\nu, \quad (10)$$

where we have included the coupling constants y_D and \bar{y}_M, \tilde{y}_M of the Dirac and the Majorana terms, respectively. We remark that the (auxiliary) flavon field Φ_3^ν does not couple to any matter field due to its $U(1)$ charge of w . It has been introduced to the model for the sole purpose of aligning the other neutrino flavons, as will be discussed in detail in Section 3.

Now that we have explicit forms for the leading order Yukawa terms associated with the up, down, charged lepton, and neutrino sectors, the next step is to break the $\Delta(96)$ flavour symmetry, and form the corresponding mass matrices, after electroweak symmetry breaking.

2.2 The Quark and Charged Lepton Mass Matrices

Just as we began the discussion of the invariant Yukawa operators under the $\Delta(96) \times U(1) \times Z_3$ flavour symmetry by writing the terms contributing to the up-type quark masses first, we begin the discussion of fermion mass matrices by considering the up sector first, as well. The flavons that must be considered to do this are Φ_2^u and $\bar{\Phi}_2^u$. They assume the following vevs (a detailed discussion of the origin of the alignment of these vevs appears in Section 3):

$$\langle \Phi_2^u \rangle = \varphi_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \langle \bar{\Phi}_2^u \rangle = \bar{\varphi}_2^u \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (11)$$

This alignment gives rise to

$$M_u \approx v_u \begin{pmatrix} y_u'' \bar{\varphi}_2^u \varphi_2^u / M^2 & 0 & 0 \\ 0 & y_u' \varphi_2^u / M & 0 \\ 0 & 0 & y_u \end{pmatrix}, \quad (12)$$

where v_u denotes the vev of the electroweak Higgs field H_u . Assuming $\varphi_2^u / M \approx \lambda^4$ and $\bar{\varphi}_2^u / M \approx \lambda^4$, where $\lambda \approx 0.225$ is the Wolfenstein parameter [1] associated with the sine of the Cabibbo angle, yields the well-known mass hierarchy among the up quarks:

$$m_u : m_c : m_t \approx \lambda^8 : \lambda^4 : 1. \quad (13)$$

Moving to the down-type quarks, the flavons Φ_2^d , Φ_3^d and $\bar{\Phi}_3^d$ need to be considered. Assume they acquire the following vacuum alignments:

$$\langle \Phi_2^d \rangle = \varphi_2^d \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \langle \Phi_3^d \rangle = \varphi_3^d \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \langle \bar{\Phi}_3^d \rangle = \bar{\varphi}_3^d \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \quad (14)$$

Adopting the left-right convention, these vevs give rise to the following down quark and charged lepton mass matrices:

$$M_d \approx v_d \begin{pmatrix} 0 & y_d''(\varphi_2^d)^2 \bar{\varphi}_3^d/M^3 & -y_d''(\varphi_2^d)^2 \bar{\varphi}_3^d/M^3 \\ y_d''(\varphi_2^d)^2 \bar{\varphi}_3^d/M^3 & y_d' \varphi_2^d \bar{\varphi}_3^d/M^2 - y_d''(\varphi_2^d)^2 \bar{\varphi}_3^d/M^3 & -y_d' \varphi_2^d \bar{\varphi}_3^d/M^2 \\ 0 & 0 & y_d \varphi_3^d/M \end{pmatrix}, \quad (15)$$

and

$$M_e \approx v_d \begin{pmatrix} 0 & y_d''(\varphi_2^d)^2 \bar{\varphi}_3^d/M^3 & 0 \\ y_d''(\varphi_2^d)^2 \bar{\varphi}_3^d/M^3 & -3y_d' \varphi_2^d \bar{\varphi}_3^d/M^2 - y_d''(\varphi_2^d)^2 \bar{\varphi}_3^d/M^3 & 0 \\ -y_d''(\varphi_2^d)^2 \bar{\varphi}_3^d/M^3 & 3y_d' \varphi_2^d \bar{\varphi}_3^d/M^2 & y_d \varphi_3^d/M \end{pmatrix}, \quad (16)$$

where v_d denotes the vev of the electroweak Higgs field, H_d . We remark that M_e is obtained from M_d by transposition and inclusion of the GJ factor of -3 [18]. The equality³ of the (12) and (21) entries of M_d , together with the vanishing (11) entry at leading order, generates the desired GST relation, *i.e.* $\theta_{12}^q \approx \theta_{12}^d \approx \sqrt{m_d/m_s}$ [17].

Assuming the following magnitudes for the flavons associated with the down quark/charged lepton sectors:

$$\varphi_2^d/M \approx \lambda, \quad \varphi_3^d/M \approx \lambda^{1+k}, \quad \bar{\varphi}_3^d/M \approx \lambda^{2+k}, \quad (17)$$

where $k = 0$ or $k = 1$, the ‘‘GUT scale’’ down quark and charged lepton mass hierarchies may be expressed as:

$$m_e/m_d = 1/3, \quad m_\mu/m_s = 3, \quad m_\tau/m_b = 1, \quad (18)$$

$$m_d : m_s : m_b \approx \lambda^4 : \lambda^2 : 1, \quad (19)$$

and the mixing angles θ_{ij}^e and θ_{ij}^d take the semi-familiar forms

$$\begin{aligned} \theta_{12}^d &\approx \lambda, & \theta_{23}^d &\approx \lambda^2, & \theta_{13}^d &\approx \lambda^3, \\ \theta_{12}^e &\approx \lambda/3, & \theta_{23}^e &\approx 0, & \theta_{13}^e &\approx 0. \end{aligned} \quad (20)$$

The GJ predictions have been scrutinised in [19] where it was shown that they are acceptable assuming particular SUSY threshold corrections, and for particular values of $\tan \beta$. We note that, for the purposes of predicting PMNS mixing angles, the only aspect of the GJ relations that will be relevant is $\theta_{12}^e \approx \lambda/3$, where such a charged lepton mixing angle can also be achieved consistently with some of the alternatives to the GJ relations proposed by Antusch and Spinrath in [19], for example those that predict $m_\mu/m_s = 9/2$.

Notice that the main difference between the cases $k = 0$ and $k = 1$ is the mass of the bottom quark and τ lepton:

$$m_b \approx m_\tau \approx \lambda^{1+k} v_d. \quad (21)$$

The two different choices for k represent two different predicted ranges for $\tan \beta$. For $k = 0$, $m_b \approx m_\tau \approx \lambda v_d$. While for $k = 1$, $m_b \approx m_\tau \approx \lambda^2 v_d$. Thus, $k = 0$ prefers a larger value of

³It is interesting to note that the (12) and (21) entries of M_d have the same sign, whereas the signs are opposite in the $S_4 \times SU(5)$ model in Ref. [20]. This is a direct consequence of $\Delta(96)$'s CG coefficients (see Appendix A).

$\tan \beta$ because $\tan \beta$ must make up for the rest of the suppression to the bottom/tau mass. Therefore, $\tan \beta \approx \lambda^{-2} \approx 25$. Yet, for $k = 1$ the flavons contribute more to the suppression of the mass and $\tan \beta \approx \lambda^{-1} \approx 5$. For the remainder of this work, we take $k = 1$. This will simplify the number of terms needed to be considered when minimising the flavon potential. It also allows for a complete listing of the schematic structures of the mass matrices in Eqs. (12), (15), and (16):

$$M_u \sim v_u \begin{pmatrix} \lambda^8 & 0 & 0 \\ 0 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_d \sim v_d \begin{pmatrix} 0 & \lambda^5 & \lambda^5 \\ \lambda^5 & \lambda^4 & \lambda^4 \\ 0 & 0 & \lambda^2 \end{pmatrix}, \quad M_e \sim v_d \begin{pmatrix} 0 & \lambda^5 & 0 \\ \lambda^5 & 3\lambda^4 & 0 \\ \lambda^5 & 3\lambda^4 & \lambda^2 \end{pmatrix}. \quad (22)$$

With the reporting of these structures, the discussion of the quark and charged lepton masses and mixings is complete. The next step is to perform similar arguments for the neutrinos, thereby calculating their masses and mixings at the GUT scale to LO.

2.3 The Neutrino Mass Matrices

The aim of this section is the calculation of the neutrino masses and mixings, associated with a $\Delta(96)$ flavour symmetry. To this end, we begin by writing the neutrino Dirac mass matrix given by the Dirac mass term in Eq. (10):

$$M_D = y_D v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (23)$$

This simple result is due to the fact that no flavons couple to the FNH_5 term (see Table 1). However, $\Phi_{\bar{3}}^\nu$ and $\Phi_{\bar{3}'}^\nu$ couple to the right-handed Majorana mass terms; in fact, without these flavons no Majorana masses would exist. The vacuum alignments of these flavon fields are determined by demanding that $\Delta(96)$ is broken to the low-energy $Z_2^S \times Z_2^U$ subgroup in the neutrino sector. Thus, their vevs must be invariant when acted upon by the corresponding Klein group generators. It is important to note that these generators take different forms for different $\Delta(96)$ representations. Their relation to the generators given in Eq. (4) can be found in Eq. (87). With this in mind, we can determine the vevs of the neutrino flavons in the $\bar{\mathbf{3}}'$ and $\bar{\mathbf{3}}$ representations which respect the desired Klein symmetry. We find

$$\langle \Phi_{\bar{3}}^\nu \rangle = \varphi_{\bar{3}}^\nu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \langle \Phi_{\bar{3}'}^\nu \rangle = \varphi_{\bar{3}'}^\nu \begin{pmatrix} v_1 \\ \frac{1}{2}(v_1 + v_3) \\ v_3 \end{pmatrix}, \quad (24)$$

where the neutrino flavon vevs $\varphi_{\bar{3}}^\nu$ and $\varphi_{\bar{3}'}^\nu$ are both assumed to be of order $\lambda^4 M$, in order to bring M (around the GUT scale) down to the seesaw scale of $\sim 10^{13}$ GeV. Then, using these invariant vevs in the contractions of the relevant irreducible representations associated with the Majorana mass terms in Eq. (10) yields the following leading order contributions to the neutrino Majorana mass matrix:

$$M_{Maj} = \bar{y}_M \varphi_{\bar{3}}^\nu \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} + \tilde{y}_M \varphi_{\bar{3}'}^\nu \begin{pmatrix} v_3 & v_1 & \frac{1}{2}(v_1 + v_3) \\ v_1 & \frac{1}{2}(v_1 + v_3) & v_3 \\ \frac{1}{2}(v_1 + v_3) & v_3 & v_1 \end{pmatrix}. \quad (25)$$

Using the mass matrices associated with the Dirac [Eq. (23)] and Majorana [Eq. (25)] mass terms, the heavy degrees of freedom associated with the right-handed neutrinos can be integrated out to generate the seesaw formula [22]:

$$m_\nu = M_D M_{Maj}^{-1} M_D^T = y_D^2 v_u^2 M_{Maj}^{-1}. \quad (26)$$

Thus, it suffices to diagonalise m_ν to reveal the light neutrino masses. This can be done with a unitary transformation such that:

$$V_\nu m_\nu V_\nu^T = m_\nu^{diag}. \quad (27)$$

This straightforward diagonalisation takes the following form:

$$m_\nu = V_\nu^\dagger \text{diag}(m_1, m_2, m_3) V_\nu^*, \quad (28)$$

where the complex masses are given by

$$m_1 = \frac{2y_D^2 v_u^2}{\sqrt{3}\bar{y}_M \varphi_{3'}^\nu (v_3 - v_1) - 6\bar{y}_M \varphi_{3'}^\nu}, \quad m_2 = \frac{2y_D^2 v_u^2}{3\bar{y}_M \varphi_{3'}^\nu (v_1 + v_3)}, \quad m_3 = \frac{2y_D^2 v_u^2}{\sqrt{3}\bar{y}_M \varphi_{3'}^\nu (v_1 - v_3) - 6\bar{y}_M \varphi_{3'}^\nu}, \quad (29)$$

and

$$V_\nu^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_+ & \frac{1}{\sqrt{3}} & a_- \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix}, \quad (30)$$

with $a_\pm = (1 \pm \frac{1}{\sqrt{3}})/2$. Up to Majorana phases and the phase matrix on the left (which will turn out to be unphysical), this is just the BT mixing form discussed in the Introduction, here applied to the neutrino sector only. It is easy to see that V_ν^\dagger translates to the following values of the neutrino mixing angle:

$$\theta_{23}^\nu = \theta_{12}^\nu = \tan^{-1}(\sqrt{3} - 1) \approx 36.2^\circ, \quad \theta_{13}^\nu = \sin^{-1}(a_-) \approx 12.2^\circ. \quad (31)$$

Before calculating the mixing matrix including the charged lepton corrections, it is worth mentioning that the complex masses in Eq. (29) do not satisfy a sum rule in general. This is due to the fact that one of the two neutrino type flavons entering in Eq. (10) depends on two free parameters, v_1 and v_3 , cf. Eq. (24). However, we will see in Section 3 that these parameters will be related via the minimisation conditions of the flavon potential. Anticipating the result of Eq. (55) we obtain $v_1 = 1$ and $v_3 = \omega^{2p}$, with $p = 1, 2$. It is easy to see that this special relation gives rise to the following new sum rule of complex neutrino masses:

$$\frac{1}{m_3} + \frac{2i(-1)^p}{m_2} - \frac{1}{m_1} = 0, \quad (32)$$

where both choices of p allow for either normal or inverted mass ordering.

2.4 Predictions for PMNS Mixing

Since the mixing matrices associated with the neutrinos and charged leptons are calculated, an approximate value for U_{PMNS} elements can be obtained by first observing that the only leading order nonzero mixing angle in the charged lepton sector is $\theta_{12}^e \approx \lambda/3$, cf. Eq. (20). In this subsection we consider the effect of such charged lepton corrections to the lepton mixing angles.

We begin with a reminder of the conventions used throughout this work. As defined in Eq. (27), $V_\nu m_\nu V_\nu^T = m_\nu^{diag}$, where the masses are still complex. The Majorana phase matrix P which renders the neutrino masses real and positive is therefore added separately in the expression for U_{PMNS} in Eq. (33). Similarly, $V_e m_e m_e^\dagger V_e^\dagger = (m_e^{diag})^2$. Thus, the left-right convention is adopted for defining the mass matrices. As a result of these conventions,

$$U_{PMNS} = V_e V_\nu^\dagger P. \quad (33)$$

As was seen in Eq. (20), $\theta_{23}^e \approx 0$, $\theta_{13}^e \approx 0$, and $\theta_{12}^e \approx \lambda/3$. This non-zero prediction for θ_{12}^e implies that V_e takes the form,

$$V_e \approx P' \begin{pmatrix} c_{12}^e & -s_{12}^e e^{-i\delta_{12}^e} & 0 \\ s_{12}^e e^{i\delta_{12}^e} & c_{12}^e & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (34)$$

where $c_{12}^e = \cos \theta_{12}^e$ and $s_{12}^e = \sin \theta_{12}^e$. δ_{12}^e is an undetermined phase and P' is a diagonal matrix consisting of three unphysical phases which may be absorbed into the charge lepton mass eigenstates.

Inserting Eqs. (30) and (34) into Eq. (33), we find,

$$U_{PMNS} \approx P'' \begin{pmatrix} a_+ c_{12}^e + \frac{1}{\sqrt{3}} s_{12}^e e^{-i\delta_{12}^e} & \frac{1}{\sqrt{3}} c_{12}^e - \frac{1}{\sqrt{3}} s_{12}^e e^{-i\delta_{12}^e} & a_- c_{12}^e - \frac{1}{\sqrt{3}} s_{12}^e e^{-i\delta_{12}^e} \\ a_+ s_{12}^e e^{i\delta_{12}^e} - \frac{1}{\sqrt{3}} c_{12}^e & \frac{1}{\sqrt{3}} s_{12}^e e^{i\delta_{12}^e} + \frac{1}{\sqrt{3}} c_{12}^e & a_- s_{12}^e e^{i\delta_{12}^e} + \frac{1}{\sqrt{3}} c_{12}^e \\ a_- & -\frac{1}{\sqrt{3}} & a_+ \end{pmatrix} P, \quad (35)$$

where we have permuted the phase matrix $\text{diag}(1, 1, -1)$ of V_ν^\dagger with the 1-2 rotation of V_e and combined both unphysical phase matrices into P'' .

Given U_{PMNS} , we identify the reactor angle from

$$\sin \theta_{13} = |(U_{PMNS})_{13}| \approx |a_- c_{12}^e - \frac{1}{\sqrt{3}} s_{12}^e e^{-i\delta_{12}^e}| \approx a_- - \frac{1}{\sqrt{3}} \theta_{12}^e \cos \delta_{12}^e. \quad (36)$$

In order to reduce the reactor angle from its BT value of 12.2° down to the values observed by Daya Bay and RENO we need to assume $\cos \delta_{12}^e \approx 1$ and hence $\delta_{12}^e \approx 0$, leading to,

$$\sin \theta_{13} \approx 0.167, \quad (37)$$

corresponding to $\theta_{13} \approx 9.6^\circ$.

With the phase $\delta_{12}^e \approx 0$ fixed by the requirement of lowering the reactor angle to an acceptable value, the rest of the angles are easy to read off the matrix, since the PMNS matrix in Eq. (35) is then real (up to Majorana phases in P) and automatically in the PDG

form with the unphysical phase matrix P'' set equal to the unit matrix. The Dirac CP phase δ is given by,

$$\delta = -\arg [(U_{PMNS})_{13}] \approx -\arg [a_- c_{12}^e - \frac{1}{\sqrt{3}} s_{12}^e] \approx 0. \quad (38)$$

The atmospheric angle is given by,

$$\tan \theta_{23} \approx \frac{\frac{1}{\sqrt{3}} c_{12}^e + a_- s_{12}^e}{a_+} \approx 0.750, \quad (39)$$

leading to $\theta_{23} \approx 36.9^\circ$, close to the uncorrected value of 36.2° . The solar angle is given by,

$$\tan \theta_{12} \approx \frac{\frac{1}{\sqrt{3}} c_{12}^e - \frac{1}{\sqrt{3}} s_{12}^e}{a_+ c_{12}^e + \frac{1}{\sqrt{3}} s_{12}^e} \approx 0.642, \quad (40)$$

leading to $\theta_{12} \approx 32.7^\circ$. It is worth noting that the phase of the solar angle correction is the same as the phase of the reactor angle correction, so both angles are nicely lowered together into the desired range.

It is convenient to express the above predictions for the solar, atmospheric and reactor angles in terms of deviation parameters (s , a and r) from TB mixing defined in Eq. (2) [9]:

$$s \approx -0.065, \quad a \approx -0.151, \quad r \approx 0.237. \quad (41)$$

These predictions for s , a and r are now in much better agreement with the 1σ ranges given in Eq. (3). This shows that, including charged lepton corrections arising from a GUT model involving the GJ and GST relations, corrects the BT predictions almost perfectly, providing that the charged lepton corrections carry a zero phase. In particular, the solar angle lies within the 1σ range, while the atmospheric angle almost falls inside the 1σ allowed interval. The predicted reactor angle of $\theta_{13} \approx 9.6^\circ$ is now well within the 2σ range. However it should also be noted that the above predictions are valid at the GUT scale and are subject to renormalisation group (RG) [23] and canonical normalisation (CN) [24] corrections. These effects were neglected since they are expected to be rather small for models with hierarchical neutrino masses, as discussed in [23, 24].

3 Vacuum Alignment

As is with flavour models of this type, the alignment of the vevs of the flavon fields must be justified by minimising a flavon potential. Thus, the explicit directions of the vevs quoted in the previous section must be derived from an explicit potential. This is the goal of the present section.

As mentioned above, we have introduced the auxiliary flavon field Φ_3^ν in order to align the other two neutrino flavons which couple to the right-handed neutrinos. Furthermore, a set of “driving fields” has to be added to generate the correct alignment of all flavon fields’ vevs of the $\Delta(96) \times SU(5)$ model. These are listed in Table 2 together with their transformation properties. Driving fields are similar to flavons in that they are gauge singlets and transform in a nontrivial way under $\Delta(96) \times U(1) \times Z_3$. However, their difference becomes manifest

Field	X_1^ν	X_2^ν	X_6^ν	X_1^d	Y_1^d	Z_1^d	X_1^u	X_1^{ud}	$X_1^{\nu d}$	X_2^{du}
$\Delta(96)$	1	2	6	1	1	1	1	1	1'	2
$U(1)$	$-6y$	$-4y$	$-2y - w$	$4y$	$-2z$	$x + 3y + z$	$2x$	$2x + 4y$	$x + 2y + 2z - w$	$2x - z$
Z_3	1	ω	ω	1	1	1	1	1	ω^2	1

Table 2: The driving fields required for the vacuum alignment of the $\Delta(96) \times SU(5) \times U(1) \times Z_3$ model. All of these fields are singlets under $SU(5)$ and have a $U(1)_R$ charge of +2.

when an additional $U(1)_R$ symmetry is introduced.⁴ Under this symmetry, the superspace variable θ is defined to have a $U(1)_R$ charge of +1. Then, all chiral supermultiplets containing SM fermions also have a $U(1)_R$ charge of +1, supermultiplets containing Higgs fields and flavons are neutral, while driving fields have a $U(1)_R$ charge of +2. A term allowed in the superpotential itself carries a $U(1)_R$ charge of +2. This implies that driving fields in the superpotential can only couple linearly to flavon fields. With the further assumption that the driving fields develop positive soft supersymmetric breaking scalar mass squared parameters at the symmetry breaking scale, the driving fields do not develop vevs. Thus, it is only necessary to enforce that the F -terms of the driving fields vanish identically. These so-called F -term conditions give rise to the vacuum alignments. In general the leading order terms of the flavon potential will be accompanied by subleading operators. Imposing the Z_3 symmetry suppresses such subleading terms, which couple the neutrino to the quark flavon fields (e.g. $X_1^u \Phi_2^u (\Phi_3^d)^2 \Phi_{3'}^\nu$ or $Y_1^d (\Phi_2^d)^2 (\Phi_3^d)^2 \Phi_{3'}^\nu$), to a negligible level. With these considerations in mind, we can begin to create and minimise the flavon potential associated with the $\Delta(96) \times SU(5)$ model.

3.1 Aligning $\langle \Phi_{\tilde{3}}^\nu \rangle$ and $\langle \Phi_{3'}^\nu \rangle$

The discussion of the alignment of the flavon fields in the $\Delta(96) \times SU(5)$ model is begun in the neutrino sector with the alignment of the vevs of the $\Phi_{\tilde{3}}^\nu$ and $\Phi_{3'}^\nu$ flavon fields. Notice that both flavons have the same charge under the $U(1)$ and Z_3 shaping symmetries of $2y$ and ω , respectively (see Table 1). In order to align both neutrino flavon fields *separately*, the structure of the Kronecker products of $\Delta(96)$ is exploited. We first discuss the flavon $\Phi_{\tilde{3}}^\nu$. Its vev is aligned using the auxiliary flavon field Φ_3^ν as well as the $\Delta(96)$ sextet driving field X_6^ν . As the $U(1)$ charge of X_6^ν involves the parameter w , its pairing with the auxiliary neutrino flavon is enforced. The product $X_6^\nu \Phi_3^\nu$ has a $U(1)$ charge of $-2y$ and could therefore couple to both neutrino flavon fields $\Phi_{\tilde{3}}^\nu$ and $\Phi_{3'}^\nu$. However, the product $\mathbf{6} \otimes \tilde{\mathbf{3}} = \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{3}' \oplus \bar{\mathbf{3}}' \oplus \mathbf{6}$ (cf. Appendix A) shows that the $\Delta(96)$ Kronecker products only allow for the coupling

$$X_6^\nu \Phi_{\tilde{3}}^\nu \Phi_{3'}^\nu. \quad (42)$$

With the driving field being a sextet under $\Delta(96)$, we obtain six F -term conditions which, using the CG coefficients of Appendix A, read

⁴ $U(1)_R$ is broken to R -parity when supersymmetry breaking terms are included.

$$\begin{aligned}
\langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3,2}^\nu \rangle + \omega^2 \langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3,3}^\nu \rangle + \omega \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3,1}^\nu \rangle &= 0, \\
\langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3,1}^\nu \rangle + \omega^2 \langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3,2}^\nu \rangle + \omega \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3,3}^\nu \rangle &= 0, \\
\langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3,3}^\nu \rangle + \omega^2 \langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3,1}^\nu \rangle + \omega \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3,2}^\nu \rangle &= 0, \\
\omega \langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3,2}^\nu \rangle + \omega^2 \langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3,3}^\nu \rangle + \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3,1}^\nu \rangle &= 0, \\
\omega \langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3,1}^\nu \rangle + \omega^2 \langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3,2}^\nu \rangle + \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3,3}^\nu \rangle &= 0, \\
\omega \langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3,3}^\nu \rangle + \omega^2 \langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3,1}^\nu \rangle + \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3,2}^\nu \rangle &= 0.
\end{aligned} \tag{43}$$

Subtracting the fourth equation down from the first, the fifth from the second, and the sixth from the third yields a set of three relations between the components of $\langle \Phi_3^\nu \rangle$ and $\langle \Phi_{3'}^\nu \rangle$:

$$\begin{aligned}
\langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3',2}^\nu \rangle &= \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3',1}^\nu \rangle, \\
\langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3',1}^\nu \rangle &= \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3',3}^\nu \rangle, \\
\langle \Phi_{3,1}^\nu \rangle \langle \Phi_{3',3}^\nu \rangle &= \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3',2}^\nu \rangle.
\end{aligned} \tag{44}$$

Similar logic can be used to further derive more relations by subtracting ω^2 multiplying the fourth, fifth and sixth lines of Eq. (43) from the first, second and third lines, respectively:

$$\begin{aligned}
\langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3',3}^\nu \rangle &= \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3',1}^\nu \rangle, \\
\langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3',2}^\nu \rangle &= \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3',3}^\nu \rangle, \\
\langle \Phi_{3,2}^\nu \rangle \langle \Phi_{3',1}^\nu \rangle &= \langle \Phi_{3,3}^\nu \rangle \langle \Phi_{3',2}^\nu \rangle.
\end{aligned} \tag{45}$$

Multiplying all three relations in Eq. (44), and separately all three relations in Eq. (45) one obtains

$$\langle \Phi_{3,1}^\nu \rangle^3 \langle \Phi_{3',1}^\nu \rangle \langle \Phi_{3',2}^\nu \rangle \langle \Phi_{3',3}^\nu \rangle = \langle \Phi_{3,2}^\nu \rangle^3 \langle \Phi_{3',1}^\nu \rangle \langle \Phi_{3',2}^\nu \rangle \langle \Phi_{3',3}^\nu \rangle = \langle \Phi_{3,3}^\nu \rangle^3 \langle \Phi_{3',1}^\nu \rangle \langle \Phi_{3',2}^\nu \rangle \langle \Phi_{3',3}^\nu \rangle. \tag{46}$$

If we now assume that none of the components of $\langle \Phi_{3'}^\nu \rangle$ vanish,⁵ we end up with

$$\langle \Phi_{3,1}^\nu \rangle^3 = \langle \Phi_{3,2}^\nu \rangle^3 = \langle \Phi_{3,3}^\nu \rangle^3, \tag{47}$$

which aligns the vev of the auxiliary flavon Φ_3^ν as

$$\langle \Phi_3^\nu \rangle \propto \begin{pmatrix} \omega^{p_1} \\ \omega^{p_2} \\ \omega^{p_3} \end{pmatrix}, \tag{48}$$

where p_1, p_2, p_3 can take the values of 0, 1, or 2. It is possible to show that Eqs. (44) and (45) only allow solutions with $p_1 + p_2 + p_3 = 0 \pmod{3}$. For the rest of this work, it is assumed that $p_1 = p_2 = p_3 = 0$.

⁵If one component of either $\langle \Phi_{3'}^\nu \rangle$ or $\langle \Phi_3^\nu \rangle$ vanishes, one can easily show by Eqs. (44) and (45) that one flavon triplet will *not* develop any vev.

The alignment of the neutrino flavon $\langle \Phi_{\bar{3}}^\nu \rangle$ is obtained by using the various relations in Eqs. (44) and (45) and the newly calculated alignment of $\langle \Phi_{\bar{3}}^\nu \rangle$. With the assumption that there are no relative phases on $\langle \Phi_{\bar{3}}^\nu \rangle$ one immediately finds the desired alignment of

$$\langle \Phi_{\bar{3}'}^\nu \rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (49)$$

Now that $\langle \Phi_{\bar{3}}^\nu \rangle$ and $\langle \Phi_{\bar{3}'}^\nu \rangle$ are aligned, it is necessary to align $\langle \Phi_{\bar{3}}^\nu \rangle$ to finish the discussion of the alignment of the neutrino flavons.

The alignment of $\langle \Phi_{\bar{3}'}^\nu \rangle$ can be accomplished by taking advantage of the $\Delta(96)$ singlets contained in the products of three neutrino flavons. Since, $\Phi_{\bar{3}}^\nu$ and $\Phi_{\bar{3}'}^\nu$ have the same $U(1)$ charge, any product of three of them will couple to the same object. From the charges in Table 2, we see that this object is X_1^ν , and the allowed flavon potential terms are⁶

$$\frac{1}{M} X_1^\nu [g_0 \Phi_{\bar{3}}^\nu \Phi_{\bar{3}}^\nu \Phi_{\bar{3}'}^\nu + g_1 \Phi_{\bar{3}}^\nu \Phi_{\bar{3}}^\nu \Phi_{\bar{3}'}^\nu + g_2 \Phi_{\bar{3}}^\nu \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu]. \quad (50)$$

With the alignment of Eq. (49), the first term, proportional to the coupling constant g_0 , vanishes identically and is therefore irrelevant for the discussion of the $\Phi_{\bar{3}'}^\nu$ vacuum. The remaining terms give rise to the following F -term condition once the solution for the already aligned $\langle \Phi_{\bar{3}'}^\nu \rangle$ is applied:

$$3g_1 (\varphi_{\bar{3}'}^\nu)^2 \left(\langle \Phi_{\bar{3}',1}^\nu \rangle + \langle \Phi_{\bar{3}',2}^\nu \rangle + \langle \Phi_{\bar{3}',3}^\nu \rangle \right) - 2g_2 \left(\langle \Phi_{\bar{3}',1}^\nu \rangle^3 + \langle \Phi_{\bar{3}',2}^\nu \rangle^3 + \langle \Phi_{\bar{3}',3}^\nu \rangle^3 - 3 \langle \Phi_{\bar{3}',1}^\nu \rangle \langle \Phi_{\bar{3}',2}^\nu \rangle \langle \Phi_{\bar{3}',3}^\nu \rangle \right) = 0. \quad (51)$$

The next step would be to solve for the conditions on the alignment of $\langle \Phi_{\bar{3}}^\nu \rangle$, but notice that there does not exist enough equations to obtain a unique solution for the alignment of $\langle \Phi_{\bar{3}}^\nu \rangle$. This leads to the introduction of the last superpotential term required for the alignment of the neutrino flavon fields' vevs:

$$X_2^\nu \Phi_{\bar{3}}^\nu \Phi_{\bar{3}'}^\nu. \quad (52)$$

As the driving field X_2^ν is a $\Delta(96)$ doublet, which does not couple to $\Phi_{\bar{3}}^\nu$, we obtain two simple F -term conditions

$$\begin{aligned} \langle \Phi_{\bar{3}',1}^\nu \rangle^2 + 2 \langle \Phi_{\bar{3}',2}^\nu \rangle \langle \Phi_{\bar{3}',3}^\nu \rangle &= 0, \\ \langle \Phi_{\bar{3}',3}^\nu \rangle^2 + 2 \langle \Phi_{\bar{3}',1}^\nu \rangle \langle \Phi_{\bar{3}',2}^\nu \rangle &= 0. \end{aligned} \quad (53)$$

With the results of Eqs. (51) and (53), there exists enough constraints to align $\langle \Phi_{\bar{3}}^\nu \rangle$ properly. Looking at Eq. (53), one finds two types of nontrivial solutions:

$$\langle \Phi_{\bar{3}'}^\nu \rangle \propto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -\frac{\omega^p}{2} \\ \omega^{2p} \end{pmatrix}, \quad (54)$$

⁶The other possible product $\Phi_{\bar{3}}^\nu \Phi_{\bar{3}'}^\nu \Phi_{\bar{3}'}^\nu$ does not contain a $\Delta(96)$ singlet.

where $p = 0, 1, 2$. Notice that the solutions with $p = 1, 2$ are of the form given in Eq. (24), hence these two solutions respect the desired low-energy Klein symmetry. With this in mind, we restrict ourselves to this set when solving Eq. (51). When this is done, four solutions are found:

$$\langle \Phi_{3'}^\nu \rangle = \pm i \varphi_{3'}^\nu \omega^{2p} \sqrt{\frac{2g_1}{3g_2}} \begin{pmatrix} 1 \\ -\frac{\omega^p}{2} \end{pmatrix}, \quad (55)$$

where $p = 1, 2$. Comparing this result and Eq. (49) to Eq. (24), it is clear to see that $\langle \Phi_{3'}^\nu \rangle$ and $\langle \Phi_{3'}^\nu \rangle$ are aligned in the proper way to spontaneously break $\Delta(96)$ to the desired low-energy Klein symmetry in the neutrino sector. With the flavons associated with the neutrino sector properly aligned, the next task is to correctly align the flavons associated with the charged leptons and quarks. This endeavour will begin with the alignment of the flavons which furnish doublet representations of $\Delta(96)$.

3.2 Aligning $\langle \Phi_2^d \rangle$, $\langle \Phi_2^u \rangle$, and $\langle \bar{\Phi}_2^u \rangle$

The quest for the correct vacuum alignment for the flavon fields is continued by considering the flavons which transform as doublets under $\Delta(96)$, *i.e.* Φ_2^d , Φ_2^u , and $\bar{\Phi}_2^u$. Considering the charges in Tables 2 and 1 for Y_1^d and Φ_2^d , respectively, allows the flavon superpotential term

$$Y_1^d \Phi_2^d \Phi_2^d = 2Y_1^d \Phi_{2,1}^d \Phi_{2,2}^d. \quad (56)$$

Clearly, the resulting F -term condition has two solutions

$$\langle \Phi_2^d \rangle \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (57)$$

of which we choose the alignment consistent with Eq. (14), *i.e.* the alignment in which $\langle \Phi_{2,2}^d \rangle$ vanishes.

Turning to the X_2^{du} driving field, we find the flavon potential coupling

$$X_2^{du} \Phi_2^d \Phi_2^u = X_{2,1}^{du} \Phi_{2,1}^d \Phi_{2,1}^u + X_{2,2}^{du} \Phi_{2,2}^d \Phi_{2,2}^u. \quad (58)$$

With the already aligned $\langle \Phi_2^d \rangle$, the F -term condition of X_2^{du} is automatically satisfied, while the condition arising from the F -term of $X_{2,1}^{du}$ enforces the alignment

$$\langle \Phi_2^u \rangle \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (59)$$

Finally, we arrive to the last doublet to align, $\langle \bar{\Phi}_2^u \rangle$. From the charges given in Tables 2 and 1, it is seen that $\bar{\Phi}_2^u$ couples to Φ_2^u and X_1^u as

$$X_1^u \Phi_2^u \bar{\Phi}_2^u = X_1^u (\Phi_{2,1}^u \bar{\Phi}_{2,2}^u + \Phi_{2,2}^u \bar{\Phi}_{2,1}^u). \quad (60)$$

Inserting the vacuum alignment for $\langle \Phi_2^u \rangle$ given in Eq. (59) immediately implies that

$$\langle \bar{\Phi}_2^u \rangle \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (61)$$

With the derivation of this last result, the vevs of all flavons transforming as a doublet of $\Delta(96)$ have been aligned accordingly. Thus, the remaining task is to align the vevs of Φ_3^d and $\bar{\Phi}_3^d$.

3.3 Aligning $\langle \Phi_3^d \rangle$ and $\langle \bar{\Phi}_3^d \rangle$

The last set of flavon fields to align is that set of flavons contributing to the down quark and charged lepton masses and mixings, Φ_3^d and $\bar{\Phi}_3^d$. This final task is begun by considering $\langle \Phi_3^d \rangle$. From Table 2, it can be seen that the leading flavon potential terms which contribute to the alignment are

$$\frac{1}{M^2} X_1^d \Phi_3^d \Phi_3^d \Phi_3^d \Phi_3^d + \frac{1}{M^3} X_1^{ud} \Phi_2^u \Phi_3^d \Phi_3^d \Phi_3^d \Phi_3^d, \quad (62)$$

which lead to the F -term conditions

$$\left(\langle \Phi_{3,2}^d \rangle^2 + 2 \langle \Phi_{3,1}^d \rangle \langle \Phi_{3,3}^d \rangle \right)^2 + 2 \left(\langle \Phi_{3,1}^d \rangle^2 + 2 \langle \Phi_{3,2}^d \rangle \langle \Phi_{3,3}^d \rangle \right) \left(\langle \Phi_{3,3}^d \rangle^2 + 2 \langle \Phi_{3,1}^d \rangle \langle \Phi_{3,2}^d \rangle \right) = 0, \quad (63)$$

and

$$\left(\langle \Phi_{3,1}^d \rangle^2 + 2 \langle \Phi_{3,2}^d \rangle \langle \Phi_{3,3}^d \rangle \right)^2 + 2 \left(\langle \Phi_{3,2}^d \rangle^2 + 2 \langle \Phi_{3,1}^d \rangle \langle \Phi_{3,3}^d \rangle \right) \left(\langle \Phi_{3,3}^d \rangle^2 + 2 \langle \Phi_{3,1}^d \rangle \langle \Phi_{3,2}^d \rangle \right) = 0. \quad (64)$$

Note that we have used the already calculated alignment of $\langle \Phi_2^u \rangle$ in order to derive Eq. (64) from the second term of Eq. (62). These two F -term conditions allow for the following sixteen vacuum alignments:

$$\langle \Phi_3^d \rangle \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \omega^{q_1} \\ 1 \\ -\frac{\omega^{2q_1}}{2} \end{pmatrix}, \begin{pmatrix} \omega^{q_2} \\ 1 \\ (-\frac{1}{2} \pm \frac{3i}{2}) \omega^{2q_2} \end{pmatrix}, \begin{pmatrix} (-2 \pm \sqrt{3}) \omega^{q_3} \\ 1 \\ (-\frac{1}{2} \pm \frac{\sqrt{3}}{2}) \omega^{2q_3} \end{pmatrix}, \quad (65)$$

where q_1 , q_2 , and q_3 take the values of 0, 1, or 2. We select the first solution in Eq. (65), as it is consistent with the assumed alignment of Eq. (14). Now that $\langle \Phi_3^d \rangle$ has been aligned there exists only one more flavon vev to align, $\langle \bar{\Phi}_3^d \rangle$.

The alignment of $\langle \bar{\Phi}_3^d \rangle$ is derived from the driving fields $X_{1'}^{\nu d}$ and Z_1^d . As can be seen from Table 2, the relevant flavon potential terms read

$$\frac{1}{M} X_{1'}^{\nu d} \Phi_3^\nu \Phi_3^d \bar{\Phi}_3^d + \frac{1}{M^2} Z_1^d \Phi_2^d \Phi_3^d \Phi_3^d \bar{\Phi}_3^d. \quad (66)$$

Inserting the already determined flavon alignments, we arrive at the F -term conditions

$$\langle \bar{\Phi}_{3,1}^d \rangle + \langle \bar{\Phi}_{3,2}^d \rangle + \langle \bar{\Phi}_{3,3}^d \rangle = 0, \quad (67)$$

$$\langle \bar{\Phi}_{3,1}^d \rangle = 0. \quad (68)$$

Therefore, the alignment of $\langle \bar{\Phi}_3^d \rangle$ is fixed uniquely as

$$\langle \bar{\Phi}_3^d \rangle \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad (69)$$

which is in agreement with the desired alignment stated in Eq. (14).

Before concluding this section, we briefly comment on the (non-)uniqueness of the achieved set of alignment vectors. The fact that the individual F -term conditions often yield multiple solutions, as seen e.g. in Eqs. (57) and (65), is a reflection of the symmetry properties of the equations: finding one solution, we can easily obtain other solutions by applying appropriate $\Delta(96)$ transformations. We have checked in each case that all obtained solutions are indeed related to each other in such a way.

Indeed, if we consider the complete flavon potential involving all flavon fields, it is generally true that a potential which gives the desired *set* of alignments, will also give any $\Delta(96)$ transformed set of alignments, *i.e.* a set of alignments where all vevs are transformed by the same arbitrary $\Delta(96)$ group element. Being symmetry transformations, all $\Delta(96)$ transformed sets of alignments are physically equivalent. However, some sets of alignments are more convenient than others when it comes to explicitly constructing a model of flavour, as done in Section 2. Therefore we are interested in generating only our desired alignments. Now, in the case of the flavon potential presented here, we face the problem that not all sets of alignments which we can generate are related by $\Delta(96)$ transformations. This means Nature could choose to fall into a vacuum which is not related to the desired set of alignments corresponding to the BT vacuum. In order to understand how much freedom Nature has, it is useful to consider the alignments of the various flavons in turn.

Starting with Eq. (57) it is clear that both solutions are related by a U transformation, *i.e.* we can choose the desired alignment of $\langle \Phi_2^d \rangle$ without loss of generality. The other two doublet flavons $\langle \Phi_2^u \rangle$ and $\langle \bar{\Phi}_2^u \rangle$ are then aligned without any ambiguities. Continuing with Eq. (65), we find sixteen solutions. One can show explicitly that these are related to the desired $\langle \Phi_3^d \rangle$ alignment by $\Delta(96)$ transformations involving an even number of U factors (and a collection of S and T factors). Such a group transformation can be used to choose the desired $\langle \Phi_3^d \rangle$ alignment, but at the same time, the same transformation must also be applied to the already aligned doublet flavon vevs. Involving an even number of U factors, one quickly finds that the form of the doublet flavons remains unaltered by such a transformation (except for possible factors of ω). Next, we consider the alignment of $\langle \Phi_3^\nu \rangle$ whose general solution is given in Eq. (48). Using a T transformation we can bring this into the standard $(1, 1, 1)^T$ form. Notice that such a T transformation does not change the above flavon alignments but only multiplies the vevs by possible factors of ω . With the choices made so far, the alignments of $\langle \Phi_3^\nu \rangle$ and $\langle \bar{\Phi}_3^d \rangle$ are determined uniquely. The ensuing set of seven alignments is obtained without loss of generality, since it is related to all other sets of the same seven alignments derived from the flavon potential terms by $\Delta(96)$ transformations. In the next and final step we have to consider the alignment of $\langle \Phi_{3'}^\nu \rangle$, cf. Eq. (54). The solution on the left is related to the three solutions on the right by $\Delta(96)$ transformations, namely $T^k U$, where $k = 0, 1, 2$. Applying such a transformation to all the other seven flavon alignments changes at least one of the them, and the desired set of alignments is not obtained any longer. Therefore, it is not possible to choose the desired alignment for $\langle \Phi_{3'}^\nu \rangle$ without loss of generality. In fact, the four solutions for the alignment of $\langle \Phi_{3'}^\nu \rangle$ generated four physically different sets of alignments which are not related by $\Delta(96)$. Two of them correspond to BT vacua, while the other two predict unphysical mixing angles. In order for our $\Delta(96)$ model to be viable, it is therefore necessary for Nature to fall into one of the two BT vacua. As the flavon potential features only four possible vacua, this is a very mild assumption.

4 The Subleading Order

In this section, we briefly discuss the next to leading order (NLO) contributions to the superpotential arising from choosing the $U(1)$ charge parameters (w, x, y , and z), in Tables 1 and 2, as unrelated. Note that for any given set of integer choices for w, x, y , and z we expect additional model-dependent NLO terms not discussed in this section. Here we only present the inevitable NLO terms resulting from such a charge assignment. By making this assumption, we can determine all minimal NLO terms allowed by the $U(1)$ and Z_3 symmetries. To start with, we ignore the $\Delta(96)$ transformation properties and find that there are two $U(1)$ neutral factors

$$A = \frac{1}{M} \bar{\Phi}_2^u, \quad B = \frac{1}{M^3} (\Phi_3^d)^2 \Phi_{3',\tilde{3}'}^\nu, \quad (70)$$

where we have introduced a unified notation for the two neutrino type flavons fields $\Phi_{3'}^\nu$ and $\Phi_{\tilde{3}'}^\nu$. A is of order λ^4 , and B of order λ^8 since we assume that the neutrino flavon vevs are of order $\lambda^4 M$. A is neutral under Z_3 while B carries charge ω . Provided the $\Delta(96)$ symmetry is satisfied one can multiply each LO term that we give by either A or B^3 . Clearly the latter is completely irrelevant as it is suppressed by λ^{24} . The former can only give corrections of order λ^4 or smaller (in case one power of A is not sufficient to generate a $\Delta(96)$ invariant term). Following the above logic and also demanding invariance under both $\Delta(96)$ and $U(1) \times Z_3$ symmetry groups, it is found that the leading NLO corrections to the Yukawa superpotential can arise from operators of the form:

$$\begin{aligned} FT_3 H_5 \frac{1}{M} \Phi_3^d A, \quad FTH_{45} \frac{1}{M^2} \bar{\Phi}_3^d \Phi_2^d A, \quad FTH_5 \frac{1}{M^3} \bar{\Phi}_3^d (\Phi_2^d)^2 A, \\ TT H_5 \frac{1}{M^2} \Phi_2^u \bar{\Phi}_2^u A, \quad T_3 T_3 H_5 A^2, \quad FN H_5 A, \quad NN \Phi_{3',\tilde{3}'}^\nu A. \end{aligned} \quad (71)$$

In order to explain the abundance of A couplings in the above set of operators, recall that having a $\bar{\Phi}_2^u \sim A$ flavon neutral under the $U(1)$ symmetry is essential for generating the up quark mass suppression in this model [cf. Eq. (8)]. Since $\bar{\Phi}_2^u \approx \lambda^4 M$, it is seen that these NLO corrections are relatively small compared to their LO counterparts and can be neglected, *i.e.* the ratio of NLO to LO Yukawa couplings for each contributing term to the Yukawa matrix is equal to or smaller than λ^4 . As an example, consider $FT_3 H_5 \frac{1}{M} \Phi_3^d A$. Regardless of the contraction, this term can only contribute to the third row (column) of M_d (M_e) at an order of λ^6 , as such also filling in the previously existing zeroes. Comparing this result to the schematic suppressions of the mass matrices given in Eq. (22), it is found that such NLO effects are negligible. Other than those terms discussed above, we also find a couple new terms not derived from the LO terms. Demanding invariance under the $U(1)$ and Z_3 symmetries they are as follows:

$$TT_3 H_5 \frac{1}{M^5} \Phi_3^d \bar{\Phi}_3^d (\Phi_2^d)^2 \Phi_{3',\tilde{3}'}^\nu B^2, \quad TT H_5 \frac{1}{M^7} (\bar{\Phi}_3^d)^2 (\Phi_2^d)^4 \Phi_{3',\tilde{3}'}^\nu B^2. \quad (72)$$

Due to the factor B^2 , these terms are certainly very suppressed and can be ignored.

A similar result is obtained when calculating the leading NLO corrections to the flavon superpotential. We find that all flavon superpotential terms receive an NLO correction

related to the coupling of A to the existing LO term. In addition, we find several new operators:

$$X_1^u \frac{1}{M^4} (\bar{\Phi}_3^d)^2 (\Phi_2^d)^4 B^3, \quad (73)$$

$$X_1^{ud} \frac{1}{M^6} (\Phi_3^d)^2 (\bar{\Phi}_3^d)^2 (\Phi_2^d)^4, \quad (74)$$

$$X_2^{du} \frac{1}{M^6} \Phi_3^d (\bar{\Phi}_3^d)^2 (\Phi_2^d)^5. \quad (75)$$

Notice that, in the above, a factor of B^3 was added to the first operator to enforce $\Delta(96)$ invariance. As a result of this, this operator is clearly heavily suppressed. Yet, the second term is of order λ^{14} which is to be compared to the LO term involving that driving field which would be of order λ^{12} . Thus, here there exists a more significant NLO term, which is related to the alignment of Φ_3^d . Finally the third term is of order λ^{13} which needs to be compared to the LO term involving that driving field, a term which is of order λ^5 . As the relative importance of the NLO term is $\frac{\lambda^{13}}{\lambda^5} = \lambda^8$, it is negligible.

In summary, the majority of the NLO to LO correction ratios are of order λ^4 (or smaller) with the exception of the correction to the flavon potential involving the driving field X_1^{ud} [cf. Eq. (74)]. As mentioned, this term contributes an F -term of order λ^{14} . Comparing to the LO term involving X_1^{ud} in Eq. (62) it is seen that the ratio of NLO to LO F -terms is suppressed by λ^2 . In principle this term could lead to significant corrections to vacuum alignment of the d-type flavons in Eq. (14), possibly filling in the zeros at order λ^2 , which would have phenomenological effects. However, in practice, by plugging in the leading order alignment into Eq. (74), such a correction vanishes. In other words the NLO correction arising from the operator in Eq. (74) is consistent with an alignment proportional to the LO result. As a consequence, there will not be any distortion of the vacuum alignment induced by this term.

5 Conclusion

Recent results from the Daya Bay and RENO Collaborations have shown that the reactor mixing angle is non-zero and quite sizeable. This result presents a new challenge for the existing paradigms of discrete flavour symmetries which attempt to describe all quark and lepton masses and mixing parameters. In the new era of a non-zero reactor angle, $\Delta(96)$ is a very promising candidate family symmetry since it is capable of predicting all the lepton mixing angles, including a quite sizeable reactor angle. However, the resulting simple Bi-trimaximal mixing pattern obtained from one embedding of the Klein symmetry (the most promising one) gives mixing angles outside the global fit ranges, and hence simple models of leptons based on $\Delta(96)$ are not viable. This motivates going beyond the simple models of leptons proposed so far, and in particular to GUT models where modest charged lepton corrections can correct the BT predictions, bringing them into agreement with the recent global fits.

In this paper we have proposed a SUSY GUT of Flavour based upon an $SU(5)$ gauge group, together with a $\Delta(96)$ flavour symmetry. In particular, we considered that the $\bar{\mathbf{5}}$ -plets of $SU(5)$ transform as a $\mathbf{3}$ under the $\Delta(96)$ flavour group. We made use of the $\mathbf{2}$

representation of $\Delta(96)$ by assigning the two **10**-plets of $SU(5)$ corresponding to the lightest two families to transform as a **2**, with the **10** associated with the third family transforming in the trivial **1** under $\Delta(96)$. In the Higgs sector, a $\overline{\mathbf{45}}$ field was added to obtain the GJ mass relations. Right-handed neutrino fields, N , transforming as singlets under $SU(5)$ and $\overline{\mathbf{3}}$ under $\Delta(96)$ were also added in order to generate neutrino masses and mixings. The family symmetry is broken by a set of eight gauge singlet flavon fields, leading to phenomenologically viable fermion mass matrices. Finally, an additional $U(1)$ and Z_3 symmetry was employed to prevent the proliferation of unwanted terms in the superpotential.

The model describes the strong mass hierarchy among the up-type quarks, $m_u : m_c : m_t \approx \lambda^8 : \lambda^4 : 1$ with no mixing in the up sector at leading order. It also reproduces the weaker down-type quark mass hierarchy $m_d : m_s : m_b \approx \lambda^4 : \lambda^2 : 1$, with quark mixing angles satisfying the GST relation (*i.e.* $\theta_{12}^q \approx \theta_{12}^d \approx \sqrt{m_d/m_s}$). The GUT-scale GJ relations $m_e/m_d = 1/3$, $m_\mu/m_s = 3$, $m_\tau/m_b = 1$ also emerge. However we emphasise that the only relevant aspect of the GJ relations for predicting the PMNS angles is the left-handed charged lepton mixing angle prediction $\theta_{12}^e \approx \lambda/3$, and this could equally well emerge from alternatives to GJ, for example those that predict $m_\mu/m_s = 9/2$.

Applying the charged lepton mixing correction $\theta_{12}^e \approx \lambda/3$ with zero phase to the approximate leading order BT values for the atmospheric, solar, and reactor mixing angles, yields $\theta_{23} \approx 36.9^\circ$, $\theta_{12} \approx 32.7^\circ$ and $\theta_{13} \approx 9.6^\circ$, respectively, at the GUT scale, in good agreement with recent global fits and leading to the prediction of a zero Dirac CP phase $\delta \approx 0$. In general, including charged lepton corrections arising from any GUT model involving $\theta_{12}^e \approx \lambda/3$, would correct the $\Delta(96)$ predictions almost perfectly, providing that the charged lepton corrections carry a zero phase. This latter feature must eventually must be explained within a more complete theory beyond the present model, where all phases are predicted, for example along the lines of the models proposed in [27].

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A Appendix: The Group Theory of $\Delta(96)$

A.1 The Structure of $\Delta(96)$

The group $\Delta(96)$ is a non-Abelian discrete subgroup of $SU(3)$ of order 96. In fact, it is the $\Delta(6n^2)$ group with $n = 4$ (see Ref. [25] for a detailed discussion of $\Delta(6n^2)$ groups). Thus,

$$\Delta(96) \cong (Z_4 \times Z_4) \rtimes S_3. \quad (76)$$

Furthermore, it has 10 conjugacy classes. They are I (the trivial conjugacy class), $3C_4$, $3C_2$, $3C'_4$, $6C'''_4$, $32C_3$, $12C''''_4$, $12C_8$, $12C'_2$, and $12C'_8$ [15, 25]. In this list of conjugacy classes, we

$\Delta(96)$	1	1'	2	3	$\tilde{\mathbf{3}}$	$\overline{\mathbf{3}}$	3'	$\tilde{\mathbf{3}}'$	$\overline{\mathbf{3}}'$	6
\mathcal{I}	1	1	2	3	3	3	3	3	3	6
$3C_4$	1	1	2	$-1 + 2i$	-1	$-1 - 2i$	$-1 + 2i$	-1	$-1 - 2i$	2
$3C_2$	1	1	2	-1	3	-1	-1	3	-1	-2
$3C_4''$	1	1	2	$-1 - 2i$	-1	$-1 + 2i$	$-1 - 2i$	-1	$-1 + 2i$	2
$6C_4''$	1	1	2	1	-1	1	1	-1	1	-2
$32C_3$	1	1	-1	0	0	0	0	0	0	0
$12C_2''$	1	-1	0	-1	-1	-1	1	1	1	0
$12C_8$	1	-1	0	i	1	$-i$	$-i$	-1	i	0
$12C_4'''$	1	-1	0	1	-1	1	-1	1	-1	0
$12C_8'$	1	-1	0	$-i$	1	i	i	-1	$-i$	0

Table 3: The Character Table of $\Delta(96)$

have adopted Schoenflies notation in which the number in front of a given conjugacy class, C_n , is the number of elements belonging to it and the subscript “ n ” denotes the order of the elements contained in it. As a result of these conjugacy classes and the theorems that prove that the number of irreducible representations is equal to the number of conjugacy classes and the sum of the squares of the dimensions of the irreducible representations is equal to the order of the group, it is easy to see that $\Delta(96)$ has two singlet irreducible representations (**1** and **1'**), one doublet irreducible representation (**2**), six triplet irreducible representations (**3**, $\tilde{\mathbf{3}}$, $\overline{\mathbf{3}}$, **3'**, $\tilde{\mathbf{3}}'$, $\overline{\mathbf{3}}'$), and one sextet (**6**) irreducible representation because

$$1+3+3+3+6+32+12+12+12+12 = 96 = 1^2+1^2+2^2+3^2+3^2+3^2+3^2+3^2+3^2+6^2. \quad (77)$$

With these irreducible representations and conjugacy classes, it is possible to write down the character table for $\Delta(96)$ by applying the logic in Ref. [25].

With the character table, it is easy to calculate the Kronecker products of $\Delta(96)$. See Table 4 for a complete listing of the Kronecker products of $\Delta(96)$. These Kronecker products will aid in the calculation of CG coefficients for the decomposition of the product representations of $\Delta(96)$. However, this cannot be done until an explicit representation (and presentation) is chosen for each of $\Delta(96)$'s irreducible representations.

A.2 Presentations of $\Delta(96)$

As discussed in the previous section, $\Delta(96) \cong (Z_4 \times Z_4) \rtimes S_3$. As a result of this, $\Delta(96)$ can be generated by four generators a , b , c and d subject to the rules [25]:

$$a^3 = b^2 = (ab)^2 = c^4 = d^4 = 1, \quad (78)$$

$$cd = dc, \quad (79)$$

$$aca^{-1} = c^{-1}d^{-1} \quad ada^{-1} = c, \quad (80)$$

$$bcb^{-1} = d^{-1} \quad bdb^{-1} = c^{-1}. \quad (81)$$

The generators (*i.e.* a , b , c , and d) along with the rules given above define a *presentation* of $\Delta(96)$. Notice that the generators a and b define the S_3 subgroup of $\Delta(96)$ whereas the

$1 \otimes \mathbf{x} = \mathbf{x}$ with \mathbf{x} any $\Delta(96)$ irrep
$1' \otimes 1' = 1$
$1' \otimes 2 = 2$
$1' \otimes \mathbf{r} = \mathbf{r}'$ when $\mathbf{r} = 3, \tilde{3}, \text{ or } \bar{3}$
$1' \otimes \mathbf{r}' = \mathbf{r}$ when $\mathbf{r} = 3, \tilde{3}, \text{ or } \bar{3}$
$1' \otimes 6 = 6$
$2 \otimes 2 = 1 \oplus 1' \oplus 2$
$2 \otimes \mathbf{r}^m = \mathbf{r} \oplus \mathbf{r}'$ when $\mathbf{r} = 3, \tilde{3}, \text{ or } \bar{3}$
$2 \otimes 6 = 6 \oplus 6$
$3^m \otimes 3^n = \tilde{3}^p \oplus \bar{3}' \oplus \bar{3}$
$3^m \otimes \tilde{3}^n = \bar{3}^p \oplus 6$
$3^m \otimes \bar{3}^n = 1^q \oplus 2 \oplus 6$
$\tilde{3}^m \otimes \tilde{3}^n = 1^q \oplus 2 \oplus \tilde{3} \oplus \tilde{3}'$
$\tilde{3}^m \otimes \bar{3}^n = 3^p \oplus 6$
$\bar{3}^m \otimes \bar{3}^n = 3 \oplus 3' \oplus \tilde{3}^p$
$3^m \otimes 6 = 3 \oplus \tilde{3} \oplus 3' \oplus \tilde{3}' \oplus 6$
$\tilde{3}^m \otimes 6 = 3 \oplus \bar{3} \oplus 3' \oplus \bar{3}' \oplus 6$
$\bar{3}^m \otimes 6 = \tilde{3} \oplus \bar{3} \oplus \tilde{3}' \oplus \bar{3}' \oplus 6$
$6 \otimes 6 = 1 \oplus 1' \oplus 2 \oplus 2 \oplus 3 \oplus 3' \oplus \tilde{3} \oplus \tilde{3}' \oplus \bar{3} \oplus \bar{3}' \oplus 6 \oplus 6$

Table 4: The Kronecker Products of $\Delta(96)$ where $m, n = 0, 1$ count the number of primes on their corresponding representation, p is equal to “ r ” if $m + n$ is even and nothing if $m + n$ is odd, and q is equal to “ r ” if $m + n$ is odd and nothing if $m + n$ is even.

generators c and d define the Abelian $Z_4 \times Z_4$ (normal) subgroup of $\Delta(96)$ [see Eq. (78) and Eq. (79)]. The other relations [Eqs. (80) and (81)] are consequences of the semidirect product. Furthermore, it is possible to relate a, b, c , and d to a smaller set of generators for $\Delta(96)$. Define these “new” generators as X and Y . The identities relating these two sets of generators (and thus presentations) for $\Delta(96)$ are given in Ref. [15] as

$$a = Y^5 X Y^4, \quad b = X Y^2 X Y^5 \quad c = X Y^2 X Y^4 \quad d = X Y^2 X Y^6. \quad (82)$$

Multiplying various combinations of a, b, c and d and their inverses yields the relations for X and Y in terms of a, b, c , and d :

$$\begin{aligned} Y &= c^{-1} b = b d, \\ Y^2 &= c^{-1} d, \\ X Y^5 &= c a^{-1}, \\ X &= (c a^{-1})(c^{-1} b)(c^{-1} d) = d^{-1} c a^{-1} b c^{-1} d. \end{aligned} \quad (83)$$

Notice that X was derived by multiplying $X Y^5$ on the right by $Y Y^2 = Y^3$ and simplified using the relations in Eqs. (78)-(81). Then, using the preceding definitions and the arithmetic in Eqs. (78)-(81), it is straightforward to show that $X Y = d d c a^{-1} d d$ and $X Y^{-1} X Y = d d a c^{-1} d d$. Then, further calculation reveals the presentation for $\Delta(96)$ given in terms of the generators X and Y put forth in Ref. [15],

$$X^2 = Y^8 = (X Y)^3 = (X Y^{-1} X Y)^3 = 1. \quad (84)$$

Now, we wish the generator XY to be diagonal. Ergo, there exists one more transformation to preform. Letting $T = XY$ and $U = X$ implies that $UT = XXY = Y$ and $XY^{-1}XY = U(UT)^{-1}UUT = UT^{-1}U^{-1}UUT = UT^{-1}UT$. Therefore, the presentation that will be used throughout the rest of this work is

$$U^2 = T^3 = (UT)^8 = (UT^{-1}UT)^3 = 1. \quad (85)$$

With this new, simpler presentation, the next task is to explicitly calculate the generators for each irreducible representation of $\Delta(96)$.

A.3 Generator Representations

In the previous section, it was shown that a faithful representation of $\Delta(96)$ can be generated by two generators, U and T , satisfying the presentation rules of Eq. (85). With this in mind, we turn to Ref. [16] which lists the generators of $\Delta(96)$ derived from Ref. [25], in a particularly useful basis for flavour model building. However, instead of listing the generators as a , b , c , and $d = bc^3b$, as both Refs. [16, 25] do, we list them in the “canonical” S , T , and U basis of S_4 (and A_4), see e.g. [26].⁷ It turns out that the analogous S_4 generators, as they are written, are not useful for $\Delta(96)$ model building, as they require a permutation of e and μ in the flavour triplet to obtain a phenomenologically viable prediction for the reactor angle. To resolve this issue, an extra 1-2 permutation is applied to the generators. This permutation only affects the U and T generators. As a result of this extra transformation, one obtains a more “natural” basis for $\Delta(96)$ model building in which the flavour triplet is the aesthetically pleasing (e, μ, τ) . However, this permutation comes at a cost, the **6** dimensional irreducible representation must be changed as well from what is given in Ref. [16]. Furthermore, we have also changed the basis of the doublet irreducible representation given in Ref. [16] to eliminate unnecessary factors of ω in the CG coefficients involving the **2**. Applying the preceding discussion to the triplet **3**, we see that

$$\begin{aligned} s_3 &= \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad u_3 = \frac{1}{3} \begin{pmatrix} -1 + \sqrt{3} & -1 - \sqrt{3} & -1 \\ -1 - \sqrt{3} & -1 & -1 + \sqrt{3} \\ -1 & -1 + \sqrt{3} & -1 - \sqrt{3} \end{pmatrix}, \\ s_3 u_3 &= \frac{1}{3} \begin{pmatrix} -1 - \sqrt{3} & \sqrt{3} - 1 & -1 \\ \sqrt{3} - 1 & -1 & -1 - \sqrt{3} \\ -1 & -1 - \sqrt{3} & \sqrt{3} - 1 \end{pmatrix}, \quad \text{and} \quad t_3 = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \end{aligned} \quad (86)$$

With these definitions for the generators of the irreducible representation **3**, we proceed by listing the generators for the other irreducible representations of $\Delta(96)$:

⁷Notice here that $S = U(UT)^4 U(UT)^4$ is not required to generate $\Delta(96)$. Yet, we list it here to draw analogy with previous works using discrete flavour symmetries.

	S	T	U
$\mathbf{1} :$	1	1	1
$\mathbf{1}' :$	1	1	-1
$\mathbf{2} :$	$\mathcal{I}_{2 \times 2}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{3} :$	s_3	t_3	u_3
$\bar{\mathbf{3}} :$	s_3	t_3^*	u_3
$\mathbf{3}' :$	s_3	t_3	$-u_3$
$\bar{\mathbf{3}}' :$	s_3	t_3^*	$-u_3$
$\tilde{\mathbf{3}} :$	$\mathcal{I}_{3 \times 3}$	t_3	vs_3
$\tilde{\mathbf{3}}' :$	$\mathcal{I}_{3 \times 3}$	t_3	$-vs_3$
$\mathbf{6} :$	$\begin{pmatrix} s_3 & 0 \\ 0 & s_3 \end{pmatrix}$	$\begin{pmatrix} t_3 & 0 \\ 0 & t_3 \end{pmatrix}$	$\begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix}$

(87)

where

$$v = - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad w = \frac{1}{3} \begin{pmatrix} 1+i & 1+i & 1-2i \\ 1+i & 1-2i & 1+i \\ 1-2i & 1+i & 1+i \end{pmatrix}. \quad (88)$$

Sometimes we shall refer to the triplet $\mathbf{3}$ generators simply as $S = s_3$, $T = t_3$, $U = u_3$ (as is done in the Introduction, for example). Notice that in Eq. (87) the “ S ” generator is an identity matrix for the representations $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\tilde{\mathbf{3}}$, and $\tilde{\mathbf{3}}'$. This is due to the aforementioned fact that these are unfaithful representations of $\Delta(96)$. These representations are unable to generate the full $\Delta(96)$ symmetry. In fact the representations $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{2}$, $\tilde{\mathbf{3}}$, and $\tilde{\mathbf{3}}'$ generate groups isomorphic to the *trivial* group, Z_2 , $S_3 \cong \Delta(6)$, $S_4 \cong \Delta(24)$, and $S_4 \cong \Delta(24)$, respectively. Yet, the urge to claim these unfaithful representations as irrelevant must be put aside when looking to calculate a complete list of CG coefficients of $\Delta(96)$, which is the next step towards understanding this group.

A.4 $\Delta(96)$ Clebsch-Gordan Coefficients

In this section, we list the CG coefficients derived from the basis given in the previous section. All CG coefficients are reported in the form $a \otimes b$, where the a_i are from the representation on the left of the product, and the b_j are from the representation on the right of the product. Notice that from $\mathbf{3} \otimes \mathbf{3}$ onward, a single set of CG coefficients yields the results for two sets of tensor product decompositions. The tensor product on the left has its representations labelled on the left, and the tensor product on the right has its representations labelled on the right. In addition to these guidelines, note that the subscripts “**s**” and “**a**” denote symmetric and anti-symmetric, respectively.

$\boxed{\mathbf{1}' \otimes \mathbf{2} = \mathbf{2}}$ $\mathbf{2} \sim \begin{pmatrix} a_1 b_1 \\ -a_1 b_2 \end{pmatrix}$	$\boxed{\mathbf{1}' \otimes \mathbf{6} = \mathbf{6}}$ $\mathbf{6} \sim \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_1 b_3 \\ -a_1 b_4 \\ -a_1 b_5 \\ -a_1 b_6 \end{pmatrix}$
$\boxed{\mathbf{1}' \otimes \mathbf{r} = \mathbf{r}' \text{ for } \mathbf{r} = \mathbf{3}, \tilde{\mathbf{3}}, \text{ or } \overline{\mathbf{3}}}$ $\mathbf{r}' \sim \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_1 b_3 \end{pmatrix}$	$\boxed{\mathbf{1}' \otimes \mathbf{r}'_1 = \mathbf{r} \text{ for } \mathbf{r} = \mathbf{3}, \tilde{\mathbf{3}}, \text{ or } \overline{\mathbf{3}}}$ $\mathbf{r} \sim \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_1 b_3 \end{pmatrix}$
$\boxed{\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_s \oplus \mathbf{1}'_a \oplus \mathbf{2}_s}$ $\mathbf{1}_s \sim a_1 b_2 + a_2 b_1$ $\mathbf{1}'_a \sim a_1 b_2 - a_2 b_1$ $\mathbf{2}_s \sim \begin{pmatrix} a_2 b_2 \\ a_1 b_1 \end{pmatrix}$	$\boxed{\mathbf{2} \otimes \mathbf{6} = \mathbf{6}_1 \oplus \mathbf{6}_2}$ $\mathbf{6}_1 \sim \begin{pmatrix} a_1 b_3 \\ a_1 b_1 \\ a_1 b_2 \\ a_2 b_5 \\ a_2 b_6 \\ a_2 b_4 \end{pmatrix}$ $\mathbf{6}_2 \sim \begin{pmatrix} a_2 b_2 \\ a_2 b_3 \\ a_2 b_1 \\ a_1 b_6 \\ a_1 b_4 \\ a_1 b_5 \end{pmatrix}$

$\mathbf{2} \otimes \mathbf{r} = \mathbf{r} \oplus \mathbf{r}' \text{ when } \mathbf{r} = \mathbf{3} \text{ or } \widetilde{\mathbf{3}}$ $\mathbf{r} \sim \begin{pmatrix} a_2 b_2 + a_1 b_3 \\ a_2 b_3 + a_1 b_1 \\ a_2 b_1 + a_1 b_2 \end{pmatrix}$ $\mathbf{r}' \sim \begin{pmatrix} a_2 b_2 - a_1 b_3 \\ a_2 b_3 - a_1 b_1 \\ a_2 b_1 - a_1 b_2 \end{pmatrix}$	$\mathbf{2} \otimes \mathbf{r}' = \mathbf{r} \oplus \mathbf{r}' \text{ when } \mathbf{r} = \mathbf{3} \text{ or } \widetilde{\mathbf{3}}$ $\mathbf{r} \sim \begin{pmatrix} a_2 b_2 - a_1 b_3 \\ a_2 b_3 - a_1 b_1 \\ a_2 b_1 - a_1 b_2 \end{pmatrix}$ $\mathbf{r}' \sim \begin{pmatrix} a_2 b_2 + a_1 b_3 \\ a_2 b_3 + a_1 b_1 \\ a_2 b_1 + a_1 b_2 \end{pmatrix}$
$\mathbf{2} \otimes \overline{\mathbf{3}} = \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}'$ $\overline{\mathbf{3}} \sim \begin{pmatrix} a_1 b_2 + a_2 b_3 \\ a_1 b_3 + a_2 b_1 \\ a_1 b_1 + a_2 b_2 \end{pmatrix}$ $\overline{\mathbf{3}}' \sim \begin{pmatrix} a_1 b_2 - a_2 b_3 \\ a_1 b_3 - a_2 b_1 \\ a_1 b_1 - a_2 b_2 \end{pmatrix}$	$\mathbf{2} \otimes \overline{\mathbf{3}}' = \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}'$ $\overline{\mathbf{3}} \sim \begin{pmatrix} a_1 b_2 - a_2 b_3 \\ a_1 b_3 - a_2 b_1 \\ a_1 b_1 - a_2 b_2 \end{pmatrix}$ $\overline{\mathbf{3}}' \sim \begin{pmatrix} a_1 b_2 + a_2 b_3 \\ a_1 b_3 + a_2 b_1 \\ a_1 b_1 + a_2 b_2 \end{pmatrix}$
$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}' \otimes \mathbf{3}' = \widetilde{\mathbf{3}}'_s \oplus \overline{\mathbf{3}}_a \oplus \overline{\mathbf{3}}'_s$ $\widetilde{\mathbf{3}}'_s \sim \begin{pmatrix} a_1 b_2 + a_2 b_1 + a_3 b_3 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 \\ a_1 b_1 + a_2 b_3 + a_3 b_2 \end{pmatrix}$ $\overline{\mathbf{3}}_a \sim \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$ $\overline{\mathbf{3}}'_s \sim \begin{pmatrix} -2a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_3 - 2a_2 b_2 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 - 2a_3 b_3 \end{pmatrix}$	$\mathbf{3}' \otimes \mathbf{3} = \widetilde{\mathbf{3}} \oplus \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}'$ $\widetilde{\mathbf{3}} \sim \begin{pmatrix} a_1 b_2 + a_2 b_1 + a_3 b_3 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 \\ a_1 b_1 + a_2 b_3 + a_3 b_2 \end{pmatrix}$ $\overline{\mathbf{3}} \sim \begin{pmatrix} -2a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_3 - 2a_2 b_2 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 - 2a_3 b_3 \end{pmatrix}$ $\overline{\mathbf{3}}' \sim \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$

$\boxed{\mathbf{3} \otimes \tilde{\mathbf{3}} = \mathbf{3}' \otimes \tilde{\mathbf{3}}' = \overline{\mathbf{3}}' \oplus \mathbf{6}}$ $\overline{\mathbf{3}}' \sim \begin{pmatrix} a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 + a_3 b_3 \end{pmatrix}$ $\mathbf{6} \sim \begin{pmatrix} a_1 b_2 + \omega^2 a_2 b_1 + \omega a_3 b_3 \\ \omega a_1 b_3 + a_2 b_2 + \omega^2 a_3 b_1 \\ \omega^2 a_1 b_1 + \omega a_2 b_3 + a_3 b_2 \\ -a_1 b_2 - \omega a_2 b_1 - \omega^2 a_3 b_3 \\ -\omega^2 a_1 b_3 - a_2 b_2 - \omega a_3 b_1 \\ -\omega a_1 b_1 - \omega^2 a_2 b_3 - a_3 b_2 \end{pmatrix}$	$\boxed{\mathbf{3} \otimes \tilde{\mathbf{3}}' = \mathbf{3}' \otimes \tilde{\mathbf{3}} = \overline{\mathbf{3}} \oplus \mathbf{6}}$ $\overline{\mathbf{3}} \sim \begin{pmatrix} a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 + a_3 b_3 \end{pmatrix}$ $\mathbf{6} \sim \begin{pmatrix} a_1 b_2 + \omega^2 a_2 b_1 + \omega a_3 b_3 \\ \omega a_1 b_3 + a_2 b_2 + \omega^2 a_3 b_1 \\ \omega^2 a_1 b_1 + \omega a_2 b_3 + a_3 b_2 \\ a_1 b_2 + \omega a_2 b_1 + \omega^2 a_3 b_3 \\ \omega^2 a_1 b_3 + a_2 b_2 + \omega a_3 b_1 \\ \omega a_1 b_1 + \omega^2 a_2 b_3 + a_3 b_2 \end{pmatrix}$
$\boxed{\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{3}' \otimes \overline{\mathbf{3}}' = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{6}}$ $\mathbf{1} \sim a_1 b_1 + a_2 b_2 + a_3 b_3$ $\mathbf{2} \sim \begin{pmatrix} a_1 b_3 + a_2 b_1 + a_3 b_2 \\ a_1 b_2 + a_2 b_3 + a_3 b_1 \end{pmatrix}$ $\mathbf{6} \sim \begin{pmatrix} a_1 b_2 + \omega^2 a_2 b_3 + \omega a_3 b_1 \\ \omega^2 a_1 b_1 + \omega a_2 b_2 + a_3 b_3 \\ \omega a_1 b_3 + a_2 b_1 + \omega^2 a_3 b_2 \\ -\omega^2 a_1 b_2 - a_2 b_3 - \omega a_3 b_1 \\ -a_1 b_1 - \omega a_2 b_2 - \omega^2 a_3 b_3 \\ -\omega a_1 b_3 - \omega^2 a_2 b_1 - a_3 b_2 \end{pmatrix}$	$\boxed{\mathbf{3} \otimes \overline{\mathbf{3}}' = \mathbf{3}' \otimes \overline{\mathbf{3}} = \mathbf{1}' \oplus \mathbf{2} \oplus \mathbf{6}}$ $\mathbf{1}' \sim a_1 b_1 + a_2 b_2 + a_3 b_3$ $\mathbf{2} \sim \begin{pmatrix} a_1 b_3 + a_2 b_1 + a_3 b_2 \\ -a_1 b_2 - a_2 b_3 - a_3 b_1 \end{pmatrix}$ $\mathbf{6} \sim \begin{pmatrix} a_1 b_2 + \omega^2 a_2 b_3 + \omega a_3 b_1 \\ \omega^2 a_1 b_1 + \omega a_2 b_2 + a_3 b_3 \\ \omega a_1 b_3 + a_2 b_1 + \omega^2 a_3 b_2 \\ \omega^2 a_1 b_2 + a_2 b_3 + \omega a_3 b_1 \\ a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \\ \omega a_1 b_3 + \omega^2 a_2 b_1 + a_3 b_2 \end{pmatrix}$
$\boxed{\tilde{\mathbf{3}} \otimes \tilde{\mathbf{3}} = \tilde{\mathbf{3}}' \otimes \tilde{\mathbf{3}}' = \mathbf{1}_s \oplus \mathbf{2}_s \oplus \tilde{\mathbf{3}}_a \oplus \tilde{\mathbf{3}}'_s}$ $\mathbf{1}_s \sim a_1 b_3 + a_2 b_2 + a_3 b_1$ $\mathbf{2}_s \sim \begin{pmatrix} a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_2 + a_2 b_1 + a_3 b_3 \end{pmatrix}$ $\tilde{\mathbf{3}}_a \sim \begin{pmatrix} a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \\ a_2 b_3 - a_3 b_2 \end{pmatrix}$ $\tilde{\mathbf{3}}'_s \sim \begin{pmatrix} a_1 b_2 + a_2 b_1 - 2a_3 b_3 \\ a_1 b_3 - 2a_2 b_2 + a_3 b_1 \\ -2a_1 b_1 + a_2 b_3 + a_3 b_2 \end{pmatrix}$	$\boxed{\tilde{\mathbf{3}}' \otimes \tilde{\mathbf{3}} = \mathbf{1}' \oplus \mathbf{2} \oplus \tilde{\mathbf{3}}' \oplus \tilde{\mathbf{3}}}$ $\mathbf{1}' \sim a_1 b_3 + a_2 b_2 + a_3 b_1$ $\mathbf{2} \sim \begin{pmatrix} a_1 b_1 + a_2 b_3 + a_3 b_2 \\ -a_1 b_2 - a_2 b_1 - a_3 b_3 \end{pmatrix}$ $\tilde{\mathbf{3}}' \sim \begin{pmatrix} a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \\ a_2 b_3 - a_3 b_2 \end{pmatrix}$ $\tilde{\mathbf{3}} \sim \begin{pmatrix} a_1 b_2 + a_2 b_1 - 2a_3 b_3 \\ a_1 b_3 - 2a_2 b_2 + a_3 b_1 \\ -2a_1 b_1 + a_2 b_3 + a_3 b_2 \end{pmatrix}$

$\boxed{\tilde{\mathbf{3}} \otimes \bar{\mathbf{3}} = \tilde{\mathbf{3}}' \otimes \bar{\mathbf{3}}' = \mathbf{3}' \oplus \mathbf{6}}$ $\mathbf{3}' \sim \begin{pmatrix} a_1 b_2 + a_2 b_3 + a_3 b_1 \\ a_1 b_1 + a_2 b_2 + a_3 b_3 \\ a_1 b_3 + a_2 b_1 + a_3 b_2 \end{pmatrix}$ $\mathbf{6} \sim \begin{pmatrix} a_1 b_2 + \omega^2 a_2 b_3 + \omega a_3 b_1 \\ a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \\ a_1 b_3 + \omega^2 a_2 b_1 + \omega a_3 b_2 \\ -\omega a_1 b_2 - \omega^2 a_2 b_3 - a_3 b_1 \\ -\omega a_1 b_1 - \omega^2 a_2 b_2 - a_3 b_3 \\ -\omega a_1 b_3 - \omega^2 a_2 b_1 - a_3 b_2 \end{pmatrix}$	$\boxed{\tilde{\mathbf{3}} \otimes \bar{\mathbf{3}}' = \tilde{\mathbf{3}}' \otimes \bar{\mathbf{3}} = \mathbf{3} \oplus \mathbf{6}}$ $\mathbf{3} \sim \begin{pmatrix} a_1 b_2 + a_2 b_3 + a_3 b_1 \\ a_1 b_1 + a_2 b_2 + a_3 b_3 \\ a_1 b_3 + a_2 b_1 + a_3 b_2 \end{pmatrix}$ $\mathbf{6} \sim \begin{pmatrix} a_1 b_2 + \omega^2 a_2 b_3 + \omega a_3 b_1 \\ a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \\ a_1 b_3 + \omega^2 a_2 b_1 + \omega a_3 b_2 \\ \omega a_1 b_2 + \omega^2 a_2 b_3 + a_3 b_1 \\ \omega a_1 b_1 + \omega^2 a_2 b_2 + a_3 b_3 \\ \omega a_1 b_3 + \omega^2 a_2 b_1 + a_3 b_2 \end{pmatrix}$
$\boxed{\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = \bar{\mathbf{3}}' \otimes \bar{\mathbf{3}}' = \mathbf{3}_a \oplus \mathbf{3}'_s \oplus \tilde{\mathbf{3}}'_s}$ $\mathbf{3}_a \sim \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$ $\mathbf{3}'_s \sim \begin{pmatrix} -2a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_3 - 2a_2 b_2 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 - 2a_3 b_3 \end{pmatrix}$ $\tilde{\mathbf{3}}'_s \sim \begin{pmatrix} a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 + a_3 b_3 \end{pmatrix}$	$\boxed{\bar{\mathbf{3}}' \otimes \bar{\mathbf{3}} = \mathbf{3}' \oplus \mathbf{3} \oplus \tilde{\mathbf{3}}}$ $\mathbf{3}' \sim \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$ $\mathbf{3} \sim \begin{pmatrix} -2a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_3 - 2a_2 b_2 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 - 2a_3 b_3 \end{pmatrix}$ $\tilde{\mathbf{3}} \sim \begin{pmatrix} a_1 b_1 + a_2 b_3 + a_3 b_2 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 \\ a_1 b_2 + a_2 b_1 + a_3 b_3 \end{pmatrix}$

$$\mathbf{3} \otimes \mathbf{6} = \mathbf{3} \oplus \tilde{\mathbf{3}} \oplus \mathbf{3}' \oplus \tilde{\mathbf{3}}' \oplus \mathbf{6}$$

$$\mathbf{3}' \otimes \mathbf{6} = \mathbf{3}' \oplus \tilde{\mathbf{3}}' \oplus \mathbf{3} \oplus \tilde{\mathbf{3}} \oplus \mathbf{6}$$

$$\mathbf{3} \longrightarrow \begin{pmatrix} a_1 b_2 - \omega^2 a_1 b_5 + \omega^2 a_2 b_1 - a_2 b_4 + \omega a_3 b_3 - \omega a_3 b_6 \\ \omega^2 a_1 b_3 - a_1 b_6 + \omega a_2 b_2 - \omega a_2 b_5 + a_3 b_1 - \omega^2 a_3 b_4 \\ \omega a_1 b_1 - \omega a_1 b_4 + a_2 b_3 - \omega^2 a_2 b_6 + \omega^2 a_3 b_2 - a_3 b_5 \end{pmatrix} \longleftarrow \mathbf{3}'$$

$$\tilde{\mathbf{3}} \longrightarrow \begin{pmatrix} a_1 b_2 - \omega^2 a_1 b_5 + a_2 b_1 - \omega^2 a_2 b_4 + a_3 b_3 - \omega^2 a_3 b_6 \\ \omega a_1 b_3 - \omega a_1 b_6 + \omega a_2 b_2 - \omega a_2 b_5 + \omega a_3 b_1 - \omega a_3 b_4 \\ \omega^2 a_1 b_1 - a_1 b_4 + \omega^2 a_3 b_2 - a_3 b_5 + \omega^2 a_2 b_3 - a_2 b_6 \end{pmatrix} \longleftarrow \tilde{\mathbf{3}}'$$

$$\mathbf{3}' \longrightarrow \begin{pmatrix} a_1 b_2 + \omega^2 a_1 b_5 + \omega^2 a_2 b_1 + a_2 b_4 + \omega a_3 b_3 + \omega a_3 b_6 \\ \omega^2 a_1 b_3 + a_1 b_6 + \omega a_2 b_2 + \omega a_2 b_5 + a_3 b_1 + \omega^2 a_3 b_4 \\ \omega a_1 b_1 + \omega a_1 b_4 + a_2 b_3 + \omega^2 a_2 b_6 + \omega^2 a_3 b_2 + a_3 b_5 \end{pmatrix} \longleftarrow \mathbf{3}$$

$$\tilde{\mathbf{3}}' \longrightarrow \begin{pmatrix} a_1 b_2 + \omega^2 a_1 b_5 + a_2 b_1 + \omega^2 a_2 b_4 + a_3 b_3 + \omega^2 a_3 b_6 \\ \omega a_1 b_3 + \omega a_1 b_6 + \omega a_2 b_2 + \omega a_2 b_5 + \omega a_3 b_1 + \omega a_3 b_4 \\ \omega^2 a_1 b_1 + a_1 b_4 + \omega^2 a_2 b_3 + a_2 b_6 + \omega^2 a_3 b_2 + a_3 b_5 \end{pmatrix} \longleftarrow \tilde{\mathbf{3}}$$

$$\mathbf{6} \sim \begin{pmatrix} a_1 b_2 + \omega a_2 b_1 + \omega^2 a_3 b_3 \\ \omega a_1 b_3 + \omega^2 a_2 b_2 + a_3 b_1 \\ \omega^2 a_1 b_1 + a_2 b_3 + \omega a_3 b_2 \\ -\omega a_1 b_5 - a_2 b_4 - \omega^2 a_3 b_6 \\ -a_1 b_6 - \omega^2 a_2 b_5 - \omega a_3 b_4 \\ -\omega^2 a_1 b_4 - \omega a_2 b_6 - a_3 b_5 \end{pmatrix} \quad \mathbf{6} \sim \begin{pmatrix} a_1 b_2 + \omega a_2 b_1 + \omega^2 a_3 b_3 \\ \omega a_1 b_3 + \omega^2 a_2 b_2 + a_3 b_1 \\ \omega^2 a_1 b_1 + a_2 b_3 + \omega a_3 b_2 \\ \omega a_1 b_5 + a_2 b_4 + \omega^2 a_3 b_6 \\ a_1 b_6 + \omega^2 a_2 b_5 + \omega a_3 b_4 \\ \omega^2 a_1 b_4 + \omega a_2 b_6 + a_3 b_5 \end{pmatrix}$$

$$\boxed{\tilde{\mathbf{3}} \otimes \mathbf{6} = \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{3}' \oplus \bar{\mathbf{3}}' \oplus \mathbf{6}}$$

$$\boxed{\tilde{\mathbf{3}}' \otimes \mathbf{6} = \mathbf{3}' \oplus \bar{\mathbf{3}}' \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{6}}$$

$$\mathbf{3} \longrightarrow \begin{pmatrix} a_1 b_2 - \omega^2 a_1 b_5 + \omega a_2 b_1 - \omega a_2 b_4 + \omega^2 a_3 b_3 - a_3 b_6 \\ a_1 b_3 - \omega^2 a_1 b_6 + \omega a_2 b_2 - \omega a_2 b_5 + \omega^2 a_3 b_1 - a_3 b_4 \\ a_1 b_1 - \omega^2 a_1 b_4 + \omega a_2 b_3 - \omega a_2 b_6 + \omega^2 a_3 b_2 - a_3 b_5 \end{pmatrix} \longleftarrow \mathbf{3}'$$

$$\bar{\mathbf{3}} \longrightarrow \begin{pmatrix} a_1 b_1 - \omega a_1 b_4 + \omega^2 a_2 b_3 - \omega^2 a_2 b_6 + \omega a_3 b_2 - a_3 b_5 \\ a_1 b_3 - \omega a_1 b_6 + \omega^2 a_2 b_2 - \omega^2 a_2 b_5 + \omega a_3 b_1 - a_3 b_4 \\ a_1 b_2 - \omega a_1 b_5 + \omega^2 a_2 b_1 - \omega^2 a_2 b_4 + \omega a_3 b_3 - a_3 b_6 \end{pmatrix} \longleftarrow \bar{\mathbf{3}}'$$

$$\mathbf{3}' \longrightarrow \begin{pmatrix} a_1 b_2 + \omega^2 a_1 b_5 + \omega a_2 b_1 + \omega a_2 b_4 + \omega^2 a_3 b_3 + a_3 b_6 \\ a_1 b_3 + \omega^2 a_1 b_6 + \omega a_2 b_2 + \omega a_2 b_5 + \omega^2 a_3 b_1 + a_3 b_4 \\ a_1 b_1 + \omega^2 a_1 b_4 + \omega a_2 b_3 + \omega a_2 b_6 + \omega^2 a_3 b_2 + a_3 b_5 \end{pmatrix} \longleftarrow \mathbf{3}$$

$$\bar{\mathbf{3}}' \longrightarrow \begin{pmatrix} a_1 b_1 + \omega a_1 b_4 + \omega^2 a_2 b_3 + \omega^2 a_2 b_6 + \omega a_3 b_2 + a_3 b_5 \\ a_1 b_3 + \omega a_1 b_6 + \omega^2 a_2 b_2 + \omega^2 a_2 b_5 + \omega a_3 b_1 + a_3 b_4 \\ a_1 b_2 + \omega a_1 b_5 + \omega^2 a_2 b_1 + \omega^2 a_2 b_4 + \omega a_3 b_3 + a_3 b_6 \end{pmatrix} \longleftarrow \bar{\mathbf{3}}$$

$$\mathbf{6} \sim \begin{pmatrix} a_1 b_5 + a_2 b_4 + a_3 b_6 \\ a_1 b_6 + a_2 b_5 + a_3 b_4 \\ a_1 b_4 + a_2 b_6 + a_3 b_5 \\ -a_1 b_2 - a_2 b_1 - a_3 b_3 \\ -a_1 b_3 - a_2 b_2 - a_3 b_1 \\ -a_1 b_1 - a_2 b_3 - a_3 b_2 \end{pmatrix} \quad \bigg| \quad \mathbf{6} \sim \begin{pmatrix} a_1 b_5 + a_2 b_4 + a_3 b_6 \\ a_1 b_6 + a_2 b_5 + a_3 b_4 \\ a_1 b_4 + a_2 b_6 + a_3 b_5 \\ a_1 b_2 + a_2 b_1 + a_3 b_3 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 \\ a_1 b_1 + a_2 b_3 + a_3 b_2 \end{pmatrix}$$

$$\overline{\mathbf{3}} \otimes \mathbf{6} = \widetilde{\mathbf{3}} \oplus \overline{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \oplus \overline{\mathbf{3}}' \oplus \mathbf{6}$$

$$\overline{\mathbf{3}}' \otimes \mathbf{6} = \widetilde{\mathbf{3}}' \oplus \overline{\mathbf{3}}' \oplus \widetilde{\mathbf{3}} \oplus \overline{\mathbf{3}} \oplus \mathbf{6}$$

$$\widetilde{\mathbf{3}} \longrightarrow \begin{pmatrix} a_1 b_3 - \omega a_1 b_6 + a_2 b_1 - \omega a_2 b_4 + a_3 b_2 - \omega a_3 b_5 \\ \omega^2 a_1 b_1 - \omega^2 a_1 b_4 + \omega^2 a_2 b_2 - \omega^2 a_2 b_5 + \omega^2 a_3 b_3 - \omega^2 a_3 b_6 \\ \omega a_1 b_2 - a_1 b_5 + \omega a_2 b_3 - a_2 b_6 + \omega a_3 b_1 - a_3 b_4 \end{pmatrix} \longleftarrow \widetilde{\mathbf{3}}'$$

$$\overline{\mathbf{3}} \longrightarrow \begin{pmatrix} a_1 b_2 - \omega^2 a_1 b_5 + \omega^2 a_2 b_3 - a_2 b_6 + \omega a_3 b_1 - \omega a_3 b_4 \\ \omega^2 a_1 b_1 - a_1 b_4 + \omega a_2 b_2 - \omega a_2 b_5 + a_3 b_3 - \omega^2 a_3 b_6 \\ \omega a_1 b_3 - \omega a_1 b_6 + a_2 b_1 - \omega^2 a_2 b_4 + \omega^2 a_3 b_2 - a_3 b_5 \end{pmatrix} \longleftarrow \overline{\mathbf{3}}'$$

$$\widetilde{\mathbf{3}}' \longrightarrow \begin{pmatrix} a_1 b_3 + \omega a_1 b_6 + a_2 b_1 + \omega a_2 b_4 + a_3 b_2 + \omega a_3 b_5 \\ \omega^2 a_1 b_1 + \omega^2 a_1 b_4 + \omega^2 a_2 b_2 + \omega^2 a_2 b_5 + \omega^2 a_3 b_3 + \omega^2 a_3 b_6 \\ \omega a_1 b_2 + a_1 b_5 + \omega a_2 b_3 + a_2 b_6 + \omega a_3 b_1 + a_3 b_4 \end{pmatrix} \longleftarrow \widetilde{\mathbf{3}}$$

$$\overline{\mathbf{3}}' \longrightarrow \begin{pmatrix} a_1 b_2 + \omega^2 a_1 b_5 + \omega^2 a_2 b_3 + a_2 b_6 + \omega a_3 b_1 + \omega a_3 b_4 \\ \omega^2 a_1 b_1 + a_1 b_4 + \omega a_2 b_2 + \omega a_2 b_5 + a_3 b_3 + \omega^2 a_3 b_6 \\ \omega a_1 b_3 + \omega a_1 b_6 + a_2 b_1 + \omega^2 a_2 b_4 + \omega^2 a_3 b_2 + a_3 b_5 \end{pmatrix} \longleftarrow \overline{\mathbf{3}}$$

$$\mathbf{6} \sim \begin{pmatrix} a_1 b_3 + \omega a_2 b_1 + \omega^2 a_3 b_2 \\ \omega^2 a_1 b_1 + a_2 b_2 + \omega a_3 b_3 \\ \omega a_1 b_2 + \omega^2 a_2 b_3 + a_3 b_1 \\ -a_1 b_6 - \omega^2 a_2 b_4 - \omega a_3 b_5 \\ -\omega a_1 b_4 - a_2 b_5 - \omega^2 a_3 b_6 \\ -\omega^2 a_1 b_5 - \omega a_2 b_6 - a_3 b_4 \end{pmatrix} \quad \bigg| \quad \mathbf{6} \sim \begin{pmatrix} a_1 b_3 + \omega a_2 b_1 + \omega^2 a_3 b_2 \\ \omega^2 a_1 b_1 + a_2 b_2 + \omega a_3 b_3 \\ \omega a_1 b_2 + \omega^2 a_2 b_3 + a_3 b_1 \\ a_1 b_6 + \omega^2 a_2 b_4 + \omega a_3 b_5 \\ \omega a_1 b_4 + a_2 b_5 + \omega^2 a_3 b_6 \\ \omega^2 a_1 b_5 + \omega a_2 b_6 + a_3 b_4 \end{pmatrix}$$

$$\mathbf{6} \otimes \mathbf{6} = \mathbf{1}_s \oplus \mathbf{1}'_a \oplus \mathbf{2}_s \oplus \mathbf{2}_a \oplus \mathbf{3}_a \oplus \tilde{\mathbf{3}}_s \oplus \bar{\mathbf{3}}_a \oplus \mathbf{3}'_s \oplus \tilde{\mathbf{3}}'_s \oplus \bar{\mathbf{3}}'_s \oplus \mathbf{6}_s \oplus \mathbf{6}_a$$

$$\mathbf{1}_s \sim a_1 b_6 + a_2 b_5 + a_3 b_4 + a_4 b_3 + a_5 b_2 + a_6 b_1$$

$$\mathbf{1}'_a \sim a_1 b_6 + a_2 b_5 + a_3 b_4 - a_4 b_3 - a_5 b_2 - a_6 b_1$$

$$\mathbf{2}_s \sim \begin{pmatrix} a_1 b_4 + a_2 b_6 + a_3 b_5 + a_4 b_1 + a_5 b_3 + a_6 b_2 \\ a_1 b_5 + a_2 b_4 + a_3 b_6 + a_4 b_2 + a_5 b_1 + a_6 b_3 \end{pmatrix}$$

$$\mathbf{2}_a \sim \begin{pmatrix} a_1 b_4 + a_2 b_6 + a_3 b_5 - a_4 b_1 - a_5 b_3 - a_6 b_2 \\ -a_1 b_5 - a_2 b_4 - a_3 b_6 + a_4 b_2 + a_5 b_1 + a_6 b_3 \end{pmatrix}$$

$$\mathbf{3}_a \sim \begin{pmatrix} a_1 b_5 + \omega^2 a_2 b_4 + \omega a_3 b_6 - \omega^2 a_4 b_2 - a_5 b_1 - \omega a_6 b_3 \\ \omega^2 a_1 b_6 + \omega a_2 b_5 + a_3 b_4 - a_4 b_3 - \omega a_5 b_2 - \omega^2 a_6 b_1 \\ \omega a_1 b_4 + a_2 b_6 + \omega^2 a_3 b_5 - \omega a_4 b_1 - \omega^2 a_5 b_3 - a_6 b_2 \end{pmatrix}$$

$$\tilde{\mathbf{3}}_s \sim \begin{pmatrix} a_1 b_2 + a_2 b_1 + a_3 b_3 - a_4 b_5 - a_5 b_4 - a_6 b_6 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 - a_4 b_6 - a_5 b_5 - a_6 b_4 \\ a_1 b_1 + a_2 b_3 + a_3 b_2 - a_4 b_4 - a_5 b_6 - a_6 b_5 \end{pmatrix}$$

$$\bar{\mathbf{3}}_a \sim \begin{pmatrix} a_1 b_4 + \omega a_2 b_6 + \omega^2 a_3 b_5 - a_4 b_1 - \omega^2 a_5 b_3 - \omega a_6 b_2 \\ \omega^2 a_1 b_6 + a_2 b_5 + \omega a_3 b_4 - \omega a_4 b_3 - a_5 b_2 - \omega^2 a_6 b_1 \\ \omega a_1 b_5 + \omega^2 a_2 b_4 + a_3 b_6 - \omega^2 a_4 b_2 - \omega a_5 b_1 - a_6 b_3 \end{pmatrix}$$

$$\mathbf{3}'_s \sim \begin{pmatrix} a_1 b_5 + \omega^2 a_2 b_4 + \omega a_3 b_6 + \omega^2 a_4 b_2 + a_5 b_1 + \omega a_6 b_3 \\ \omega^2 a_1 b_6 + \omega a_2 b_5 + a_3 b_4 + a_4 b_3 + \omega a_5 b_2 + \omega^2 a_6 b_1 \\ \omega a_1 b_4 + a_2 b_6 + \omega^2 a_3 b_5 + \omega a_4 b_1 + \omega^2 a_5 b_3 + a_6 b_2 \end{pmatrix}$$

$$\tilde{\mathbf{3}}'_s \sim \begin{pmatrix} a_1 b_2 + a_2 b_1 + a_3 b_3 + a_4 b_5 + a_5 b_4 + a_6 b_6 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_6 + a_5 b_5 + a_6 b_4 \\ a_1 b_1 + a_2 b_3 + a_3 b_2 + a_4 b_4 + a_5 b_6 + a_6 b_5 \end{pmatrix}$$

$$\bar{\mathbf{3}}'_s \sim \begin{pmatrix} a_1 b_4 + \omega a_2 b_6 + \omega^2 a_3 b_5 + a_4 b_1 + \omega^2 a_5 b_3 + \omega a_6 b_2 \\ \omega^2 a_1 b_6 + a_2 b_5 + \omega a_3 b_4 + \omega a_4 b_3 + a_5 b_2 + \omega^2 a_6 b_1 \\ \omega a_1 b_5 + \omega^2 a_2 b_4 + a_3 b_6 + \omega^2 a_4 b_2 + \omega a_5 b_1 + a_6 b_3 \end{pmatrix}$$

$$\mathbf{6}_s \sim \begin{pmatrix} a_4 b_5 + a_5 b_4 - 2a_6 b_6 \\ a_4 b_6 - 2a_5 b_5 + a_6 b_4 \\ -2a_4 b_4 + a_5 b_6 + a_6 b_5 \\ -a_1 b_2 - a_2 b_1 + 2a_3 b_3 \\ -a_1 b_3 + 2a_2 b_2 - a_3 b_1 \\ 2a_1 b_1 - a_2 b_3 - a_3 b_2 \end{pmatrix} \quad \mathbf{6}_a \sim \begin{pmatrix} a_4 b_5 - a_5 b_4 \\ a_6 b_4 - a_4 b_6 \\ a_5 b_6 - a_6 b_5 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \\ a_2 b_3 - a_3 b_2 \end{pmatrix}$$

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